स्वाध्याय स्वमन्थन स्वावलम्बन

UTTAR PRADESH RAJARSHI TANDON OPEN UNIVERSITY

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UGMM-01
Calculus

FIRST BLOCK
Elements of Differential Calculus

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CALCULUS

Block

1

ELEMENTS OF DIFFERENTIAL CALCULUS

CALCULUS

This course on calculus is being offered to people entering the Bachelor's Degree Programme. The reason for this is the relevance of calculus to the varied problems facing mankind. Derivatives and integrals, the two basic tools of calculus have proved very useful in solving a multitude of problems in many different academic disciplines. Now-a-days calculus is being used in building abstract models for the study of population ecology, cybernetics, management practices, economics and medicine, apart from its well-known applications in physics. It is this immensely applicable nature of calculus which demands that we become familiar with its basic concepts.

In this course we shall be dealing with ideas that have evolved over hundreds of years and that were formalised by the greatest geniuses of all time. Of course, we will not study the subject as it originated, but will take advantage of the improvements made in calculus over the years. As you read this course further, you will realise that the English mathematician Isaac Newton (1642-1727) and the German mathematician Gottfried Wilhelm Leibniz(1646-1716) were the major contributors to the development of calculus. We will also have a chance to look at the contributions of some other mathematicians like Lagrange, Taylor and Maclaurin, to name a few.

This course is divided into four blocks. In the first block we shall review some fundamental concepts of the real number system and functions. We shall also study the concepts of derivatives of functions in this block. In the second block we shall study how derivatives help us to get information about various geometrical properties of curves. The third block will introduce you to the second important concept : that of integrals. And we shall be reading about the applications of calculus in the last block. We have prepared a video programme entitled "Curves" based on the material in Bolck 2. The video cassette will be available at your Study Centre

Throughout this course our main emphasis will be on techniques rather than on theory. So we have not included many proofs here. You will be able to find the proofs of many of these theorems in the course on real analysis. A word of friendly advice here. To master the various techniques presented in this course, you will need to put in a lot of practice. You should attempt all the exercises as you go along. In addition, you shoould look up some other books in the library of your Study Centre, and try to solve some exercises from these books too.

We have left some blank space after each exercise. You should use this space for solving the exercise.

Some Useful Books

1 Differential Calculus by Shanti Narayan.

2 Integral Calculus by Shanti Narayan.

NOTATIONS AND SYMBOLS

{x : x satisfies property P}	The set of all x such that x satisfies property P.
$\in$	belongs to
$\notin$	does not belong to
$\subseteq$ ($\subset$)	is contained in (is properly contained in)
$\not\subset$	is not contained in
$A \cup B$	The union of the sets A and B
$A \cap B$	The intersection of the sets A and B
$A \setminus B$	A complement B
N	The set of natural numbers
Z	The set of integers
Q	The set of rational numbers
R	The set of real numbers
C	The set of complex numbers
$\Rightarrow$	implies
$\Leftrightarrow$	implies and is implied by
iff	if and only if
$<(\leq)$	is less than (is less than or equal to)
$>(\geq)$	is greater than (is greater than or equal to)
$\exists$	there exists
$\forall$	for all
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots + a_n$
$\therefore$	therefore
i.e.	that is
w.r.t.	with respect to
$x \to a$	x tends to a
$\lim_{x \to a} f(x)$	limit of f(x) as x tends to a
$f : X \to Y$	f is a function from X to Y
$x \mapsto f(x)$	a function f taking x to f(x)
$\frac{dy}{dx}$, y_1	derivative of y w.r.t. x
f (x)	derivative of f(x) w.r.t. x
$f'(x)\big\|_{x=a}$	derivative of f(x) w.r.t. x at x = a
n!	factorial n = n(n − 1) 3.2.1
C(n, r)	the number of combinations of r things taken out of n = $\frac{n!}{r!(n-r)!}$
$\simeq$	is approximately equal to
max {x, y}	the maximum of x and y
min {x, y}	the minimum of x and y.

BLOCK 1 ELEMENTS OF DIFFERENTIAL CALCULUS

This is the first of the four blocks which you will be studying for the calculus course. In this block we shall be dealing mainly with the concept of differentiation and the various methods of finding derivatives. To fully appreciate the concept of a derivative, you will need to be familiar with the notion of the limit of a function. You can find a discussion of limits in Unit 2.

We suppose that you are already familiar with functions. But, just to refresh your memory, we have given a brief account of functions in Unit 1. In Unit 1 we also recall several properties of the real number system, which we shall be using, explicitly or implicitly, in the rest of the course. It is also possible that some of you have not studied some aspects of the real number system and functions earlier. In that case Unit 1 will help you prepare a firm ground for the imposing structure of calculus which follows.

We shall introduce the concept of a derivative in Unit 3, and then acquaint you with the derivatives of some standard functions in Units 4 and 5.

In Units 2 to 5, we have included a number of examples. Please go through them carefully. They will help you in a better understanding of the concepts discussed and will also serve as a guide in solving the exercises.

UNIT 1 REAL NUMBERS AND FUNCTIONS

Structure

1.1 INTRODUCTION

This is the first unit of the course on Calculus. We thought it would be a good idea to acquaint you with some basic results about the real number system and functions, before you actually start your study of Calculus. Perhaps, you are already familiar with these results. But, a quick look through the pages will help you in refreshing your memory, and you will be ready to tackle the course.

In the next three sections of this unit, we shall present some results about the real number system. You will find a number of examples of various types of functions in Sections 5 to 7. You should also study the graphs of these functions carefully. It is very important to be able to visualise a given function. In fact, try to draw a graph whenever you encounter a new function. We shall systematically study the tracing of curves in Unit 9.

Objectives

After reading this unit you should be able to:

- recall the basic properties of real numbers,
- derive other properties with the help of the basic ones,
- identify various types of bounded and unbounded intervals,
- define a function and examine whether a given function is one-one/onto,
- investigate whether a given function has an inverse or not,
- define the scalar multiple, absolute value, sum, difference, product, quotient of the given functions,
- determine whether a given function is even, odd, monotonic or periodic.

1.2 BASIC PROPERTIES OF R

In the next three sections, we are going to tell you about the set of real numbers which is all-pervading in mathematics. The real number system is the foundation on which a large part of mathematics, including calculus, rests. Thus, before we actually start learning calculus, it is necessary to understand the structure of the real number system.

You are already familiar with the operations of addition, subtraction, multiplication and division of real numbers, and with inequalities. Here we shall quickly recall some of their properties. We start with the operation of addition:

A1 R is closed under addition.
If x and y are real numbers, then $x + y$ is a unique real number.

A2 Addition is associative.
$x + (y + z) = (x + y) + z$ holds for all x, y, z in **R**.

A3 Zero exists.
There is a real number 0 such that
$x + 0 = 0 + x = x$ for all x in **R**.

A4 Negatives exist.
For each real number x, there exists a real number y (called a negative or an additive inverse of x, and denoted by $-x$) such that $x + y = y + x = 0$.

A5 Addition is commutative.
$x + y = y + x$ holds for all x, y in **R**.

Similar to these properties of addition, we can also list some properties of the operation of multiplication:

M1 R is closed under multiplication.
If x and y are real numbers, then $x.y$ is a unique real number.

M2 Multiplication is associative.
$x . (y . z) = (x . y).z$ holds for all x, y, z in **R**.

M3 Unit element exists.
There exists a real number 1 such that
$x . 1 = 1 . x = x$ for every x in **R**.

M4 Inverses exist.
For each real number x other than 0, there exists a real number y (called a multiplicative inverse of x and denoted by x^{-1}, or by $1/x$) such that
$xy = yx = 1$.

M5 Multiplication is commutative.
$xy = yx$ holds for all x, y in **R**.
The next property involves addition as well as multiplication.

D Multiplication is distributive over addition.
$x(y + z) = xy + xz$ holds for all x, y, z in **R**

You may have come across a "field" in the course on Linear Algebra.

Remark 1: The fact that the above eleven properties are satisfied is often expressed by saying that the real numbers form a *field* with respect to the usual addition and multiplication operations.

In addition to the above mentioned properties, we can also define an order relation on **R** with the help of which we can compare any two real numbers. We write $x > y$ to mean that x is greater than y. The order relation '>' has the following properties:

O1 Law of Trichotomy holds.
For any two real numbers a, b, one and only one of the following holds:
$a > b$, $a = b$, $b > a$.

O2 '>' is transitive.
If $a > b$ and $b > c$, then $a > c$, $\forall a, b, c \in$ **R**.

O3 Addition is monotone.
If a, b, c, in **R** are such that $a > b$, then $a + c > b + c$.

O4 Multiplication is monotone in the following sense.
If a, b, c in **R** are such that $a > b$ and $c > 0$, then $ac > bc$.
Caution: $a > b$ and $c < 0 \Rightarrow ac < bc$.

Remark 2: Any field together with a relation > satisfying O1 to O4 is called an **ordered field**. Thus **R** with the usual > is an example of an ordered field.

Notations: We write $x < y$ (and read x is less than y) to mean $y > x$. We write $x \leq y$ (and read x is less than or equal to y) to mean either $x < y$ or $x = y$. We write $x \geq y$ (and read x is greater than or equal to y) if either $x > y$ or $x = y$.

A number x is said to be positive or negative according as $x > 0$ or $x < 0$. If $x \geq 0$, we say that x is non-negative.

Now, you know that given any number $x \in$ **R**, we can always find a number $y \in$ **R** such that $y \geq x$. (In fact, there are infinitely many such real numbers y). Let us see what happens when we take any sub-set of R instead of a single real number x. Do you think that, given a set $S \subseteq R$, it is possible to find $u \in R$ such that $u \geq x$ for all $x \in S$?

Before we try to answer this question, let us look at a definition.

Definition 1 Let S be a subset of **R**. An element u in **R** is said to be an **upper bound** of S if $u \geq x$ holds for every x in S. We say that S is **bounded above**, if there is an upper bound of S. for S

Now we can reword our earlier questions as follows : Is it possible to find an upper bound for a given set ?

Let us consider the set $Z^- = \{-1, -2, -3, -4, ...\}$

Now, each $x \in Z^-$ is negative. Or, in other words, $x < 0$ for all $x \in Z$. So you see, in this case we are able to find an upper bound, namely zero, for our set Z^-

On the other hand, if we consider the set of natural numbers, $N = \{1, 2, 3 ....\}$, obviously we will not be able to find an upper bound. Thus N is not bounded above.

You will, of course, realise that if u is an upper bound for a set S then $u + 1$, $u + 2$, $u + 3$, ..., (in fact, $u+r$, where r is any positive number) are all upper bounds of S. For example, we have seen that 0 is an upper bound for Z^-. Check that 1, 2, 3, 4.8, ... are all upper bounds of Z^-.

From among all the upper bounds of a set S, which is bounded above, we can choose an upper bound u such that u is less than or equal to every upper bound of S. We call this u the **least upper bound or the supremum of S**. For example, consider the set

$T = \{x \in R : x^2 \leq 4\} = \{x \in R : -2 \leq x < 2\}$

Now 2, 3, 3.5, 4, $4 + \pi$ are all upper bounds for this set.

But you will see that 2 is less than any other upper bound.

Hence 2 is the supremum or the least upper bound of T.

You will agree that -1 is the l.u.b. (least upper bound) of Z^-.

Note that for both the sets T and Z^-, the l.u.b. belonged to the set. This may not be true in general. Consider the set of all negative real numbers $R^- = \{x : x < 0\}$. The l.u.b. of this set is 0. But $0 \notin R^-$.

Working on similar lines we can also define a **lower bound** for a given set S to be a real number v such that $v \leq x$ for all $x \in S$. We shall say that a set is bounded below, if we can find a lower bound for it. Further, the lower bound of S which is greater than any other lower bound of S will be called its **infimum or greatest lower bound** (g.l.b.).

As in the case of l.u.b., remember that the g.l.b. of a set may or may not belong to the set.

We shall say that a set $S \subset R$ is bounded if it has both an upper bound and a lower bound.

Based on this discussion you will be able to solve the following exercise.

E 1) Give examples to illustrate the following:

a) A set of real numbers having a lower bound,

b) A set of real numbers without any lower bound,

c) A set of real numbers whose g.l.b. does not belong to it,

d) A bounded set of real numbers.

Now we are ready to state an important property of R.

C The order is complete.

Every non-empty subset S of R that is bounded above, has a supremum. (We shall use this property in Unit 10).

Many more properties are either restatements or consequences of these sixteen properties. Here is a list of some of them.

1 Zero is unique.
If $x + 0' = x$ for all x in R, then $0' = 0$.

2 Additive inverse is unique.
For each x in R, there is a unique y in R such that $x + y = y + x = 0$.

3 Addition is cancellative.
If $x + y = x + z$, then $y = z$.

4 Unity is unique.
If $x.1' = x$ for all x in R, then $1' = 1$.

5 Multiplicative inverse is unique.
For each non-zero real number x, there is a unique y in R such that $xy = yx = 1$.

6 Multiplication is cancellative.
If $xy = xz$ and $x \neq 0$, then $y = z$.

Definition 2: If x and y are any two real numbers, the result of subtraction of y from x is denoted by $x - y$ and is defined as $x + (-y)$. Similarly, the division $x \div y$ (also denoted by x/y) is defined as xy^{-1}, provided $y \neq 0$.

Now we are ready to list a few more properties. You are already aware of these. But let us quickly recall them.

7 $-(x + y) = (-x) + (-y)$ for all x, y in R.

8 If $xy = 0$, then either $x = 0$ or $y = 0$.

9 $(x^{-1})^{-1} = x$ for all $x \neq 0$ in R.

10 If x and y are non zero numbers such that $x^{-1} = y^{-1}$, then $x = y$.

11 If $a < b$ and $c > 0$, then $ac < bc$.

12 a is positive if and only if $-a$ is negative.

13 If $a < b$ and $c < d$, then $a + c < b + d$.

14 If $a > b$ and $c < 0$, then $ac < bc$.

15 a^2 is non-negative for all a in R.

16 If a and b are positive, then

i) $a^2 = b^2 \Leftrightarrow a = b$.

(The symbol $\Leftrightarrow$ is read as 'if and only if')

ii) $a^2 > b^2 \Leftrightarrow a > b$

iii) $a^2 < b^2 \Leftrightarrow a < b$

17 If $b > 0$, then $a^2 < b^2 \Leftrightarrow -b < a < b$.

You are also familiar with the following subsets of R:

1) The set N of natural numbers. Note that it is the smallest subset of R possessing the following properties:

i) $1 \in N$

ii) $k \in N \Rightarrow k + 1 \in N$

2) The set Z of integers. It is the smallest subset of R possessing the following properties:
   i) $Z \supset N$
   ii) If $x, y \in Z$, then $x - y \in Z$.
3) The set Q of rational numbers. We observe that it is the smallest subset of R possessing the following properties:
   i) $Q \supset Z$
   ii) If $x, y \in Q$ and $y \neq 0$, then $xy^{-1} \in Q$.

You must have also studied the following properties of these sets.

1) $k \in N$ if and only if k is a positive integer, that is, $k \in Z$ and $k > 0$.
2) The operations of addition and multiplication on N satisfy A1, A2, A5, M1, M2, M3, M5 and D. They do not, however, satisfy A3, A4 and M4.
3) The operations on Q satisfy A1 to A5, M1 to M5 and 01 to 04. Therefore Q is an ordered field. But C is not satisfied, that is, Q is not order- complete.

We list here some more properties of these sets which you will find useful in your study of calculus:

4) **Archimedean Property:** If a and b are any real numbers and if $b > 0$, then there is a positive integer n such that $nb > a$.
5) If a is any real number, there is a positive integer n such that $n > a$ (Archimedean property applied to a and 1).
6) A real number s is the supremum of a set $S \subset R$ if and only if the following conditions are satisfied.
   i) $s \geq x$ for all x in S.
   ii) For each $\varepsilon > 0$, there is a y in S such that $y > s - \varepsilon$.

   For example, consider the set $A = \{x \in R : 8 \leq x < 10\}$. 10 is the supremum of this set. Now, if we are given any ε, say, $\varepsilon = 0.01$, we should be able to find some $y \in A$ such that $y > 10 - 0.01 = 9.99$. As you can see, $y = 9.999$ serves our purpose.

   Now 10.01 is also an upper bound for A. But 10.01 is not the supremum of A. For $\varepsilon = 0.001$, we cannot find any $y \in A$ such that $y > 10.01 - 0.001 = 10.009$.
7) Every nonempty set of real numbers that is bounded below, has an infimum.

ε (epsilon) is a Greek letter used to denote small real numbers.

The exercise below can now be done easily.

E E2) a) Show that the set of positive real numbers is bounded below. What is its infimum?

b) Write the characterisation of the infimum of a subset of R, which corresponds to 6) above. Give an example.

1.3 ABSOLUTE VALUE

In this section we shall define the absolute value of a real number. You will realise the importance of this simple concept as you study the later units.

Definition 3: If x is a real number, its absolute value, denoted by $|x|$ (read as **modulus of** x, or mod x), is defined by the following rules:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For example, we get

$|5| = 5$, $|-5| = 5$,

$|1.7| = 1.7$, $|-2| = 2$, $|0| = 0$

It is obvious that $|x|$ is defined for all $x \in \mathbf{R}$. The following theorem gives some of the important properties of $|x|$.

Theorem 1: If x and y be any real numbers, then

a) $|x| = \max\{-x, x\}$

b) $|x| = |-x|$

c) $|x|^2 = x^2 = |-x|^2$

d) $|x + y| \leq |x| + |y|$ **(the triangle inequality)**

e) $|x - y| \geq \big||x| - |y|\big|$

Proof:

a) By the law of trichotomy (01) applied to the real numbers x and 0, exactly one of the following holds:
i) $x > 0$, ii) $x = 0$, or iii) $x < 0$.

Let us consider these one by one.

i) If $x > 0$, then $|x| = x$ and $x > -x$, so that
$\max\{-x, x\} = x$ and hence $|x| = \max\{-x, x\}$

ii) If $x = 0$, then $x = 0 = -x$, and therefore,
$\max\{-x, x\} = 0$. Also $|x| = 0$, so that $|x| = \max\{-x, x\}$

iii) If $x < 0$, then $|x| = -x$, and $-x > x$, so that
$\max\{-x, x\} = -x$. Thus, again, $|x| = \max\{-x, x\}$.

From this it follows that $x \leq |x|$

b) $|-x| = \max\{-(-x), -x\} = \max\{x, -x\} = \max\{-x, x\} = |x|$.

c) If $x \geq 0$, then $|x| = x$, so that $|x|^2 = x^2$.
If $x < 0$, then $|x| = -x$, so that $|x|^2 = (-x)^2 = x^2$.
Therefore, for all $x \in \mathbf{R}$, $|x|^2 = x^2$.
Also $|-x|^2 = |x|^2$, because $|-x| = |x|$ by (b). Thus, we have $|x|^2 = x^2$

d) We shall consider two different cases according as
i) $x + y \geq 0$ or ii) $x + y < 0$.
Let $x + y \geq 0$. Then $|x + y| = x + y$. Now $x \leq |x|$ and $y \leq |y|$ by (a). Therefore.
$|x + y| = x + y \leq |x| + |y|$.
Let $x + y < 0$. Then $-(x + y) > 0$, that is,
$(-x) + (-y) > 0$ and we can use the result of (i)
for $-x$ and $-y$. Now $|x + y| = |-(x + y)|$ by (b).

Thus, $|x + y| = |(-x) + (-y)| \leq |-x| + |-y|$, by i)
$= |x| + |y|$, by (b).

Therefore, we get $|x + y| \leq |x| + |y|$.

Thus we find that for all $x, y \in \mathbf{R}$, $|x + y| \leq |x| + |y|$.

e) By writing $x = (x - y) + y$ and applying the triangle inequality to the numbers $x - y$ and y, we have

$|x| = |(x - y) + y| \leq |x - y| + |y|$,

so that $|x| - |y| \leq |x - y|$. ... (1)

Since (1) holds for all x and y in $\mathbf{R}$, therefore by interchanging x and y in (1), we have

$|y|-|x| \leq |y-x| = |-(x-y)| = |x-y|$,

so that $-(|x|-|y|) \leq |x-y|$. ... (2)

From (1) and (2) we find that $|x|-|y|$ and its negative $-(|x|-|y|)$ are both less than or at the most equal to $|x-y|$. Therefore, $\max\{|x|-|y|, -(|x|-|y|)\} \leq |x-y|$.

But the left hand side of the above inequality is simply $||x|-|y||$. Therefore, we have

$||x|-|y|| \leq |x-y|$

That is, $|x-y| \geq ||x|-|y||$ for all $x, y \in \mathbf{R}$.

Now you should be able to prove some easy consequences of this theorem. The following exercise will also give you some practice in manipulating absolute values. This practice will come in handy when you study Unit 2.

E 3) Prove the following:

a) $x = 0 \Leftrightarrow |x| = 0$

b) $|xy| = |x| \cdot |y|$

c) $|1/x| = 1/|x|$, if $x \neq 0$

d) $|x-y| \leq |x|+|y|$

e) $|x+y+z| \leq |x|+|y|+|z|$

f) $|xyz| = |x| \cdot |y| \cdot |z|$

e) and f) can be extended to any number of reals. Now if $a \in \mathbf{R}$ and $\delta > 0$, then $|x-a| < \delta \Rightarrow x-a < \delta$, or $-(x-a) < \delta$ according as $x-a \geq 0$, or $x-a < 0$, respectively.

If $x-a < \delta$, this means that $x < a+\delta$

If $-(x-a) < \delta$, this means that $a-\delta < x$.

Thus, we get that $|x-a| < \delta \Rightarrow a-\delta < x < a+\delta$.

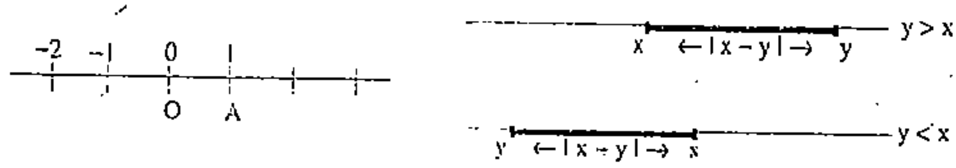
1.4 INTERVALS ON THE REAL LINE

Before we define an interval let us see what is meant by a number line. The real numbers in the set R can be put into one-to-one correspondence with the points on a straight line L. In other words, we shall associate a unique point on L to each real number and vice versa.

Consider a straight line L (see Fig. 1(a)). Mark a point O on it. The point O divides the straight line into two parts. We shall use the part to the left of O for representing negative real numbers and the part to the right of O for representing positive real numbers. We choose a point A on L which is to the right of O. We shall represent the number 0 by O and 1 by A. OA can now serve as a unit. To each positive real number x we can associate exactly one point P lying to the right of O on L, so that OP = | x | units (= x units). A negative real number y will be represented by a point Q lying to the left of O on the straight line L, so that OQ = | y | units (= −y units, since y is negative). We thus find that to each real number we can associate a point on the line. Also, each point S on the line represents a unique real number z, such that | z | = OS. Further, z is positive if S is to the right of O, and is negative if S is to the left of O.

Distance is always non-negative.

This representation of real numbers by points on a straight line is often very useful. Because of this one-to-one correspondence between real numbers and the points of a straight line, we often call a real number " a point of R". Similarly L is called a "number line". Note that the absolute value or the modulus of any number x is nothing but its distance from the point O on the number line. In the same way, | x − y | denotes the distance between the two numbers x and y (see Fig. 1(b)).



(a) (b)

Figure 1 : (a) Number line b) Distance between x and y is | x − y |

Now let us consider the set of all real numbers which lie between two given real numbers a and b. Actually, there will be four different sets satisfying this loose condition.

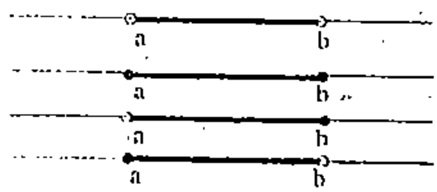
These are:

i)]a, b[= {x : a < x < b}

ii) [a, b] = {x : a ≤ x ≤ b}

iii)]a, b] = {x : a < x ≤ b}

iv) [a, b[= {x : a ≤ x < b}



The representation of each of these sets is given alongside. Each of these sets is called an **interval**, and a and b are called the end points of the interval. The interval]a, b[, in which the end points are not included, is called an **open interval**. Note that in this case we have drawn a hollow circle around a and b to indicate that they are not included in the graph. The set [a, b] contains both its end points and is called a **closed interval**. In the representation of this closed interval, we have put thick black dots at a and b to indicate that they are included in the set.

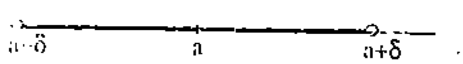
The sets [a, b[and]a,b] are called **half-open (or half-closed) intervals or semi-open (or semi-closed) intervals**, as they contain only one end point. This fact is also indicated in their geometrical representation.

If a = b,]a, a [=] a, a] = [a, a[= ∅ and [a, a] = a.

Each of these intervals is bounded above by b and bounded below by a.

Can we represent the set I = {x : | x − a | < δ} on the number line? Yes, we can. We know that | x − a | can be thought of as the distance between x and a. This means I is the set of all numbers x, whose distance from a is less than δ. Thus,

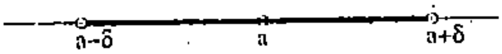
I = {x : | x − a | < δ}



is the open interval $]a-\delta, a+\delta[$. Similarly, $I_1 = \{x : |x-a| \le \delta\}$ is the closed interval $[a-\delta, a+\delta]$. Sometimes we also come across sets like $I_2 = \{x : 0 < |x-a| < \delta\}$. This means if $x \in I_2$, then the distance between x and a is less than δ, but is not zero. We can also say that the distance between x and a is less than δ, but $x \ne a$. Thus,

$I_2 =]a-\delta, a+\delta[\setminus \{a\}$

$=]a-\delta, a[\cup]a, a+\delta[$.



Apart from the four types of intervals listed above, there are a few more types. These are:

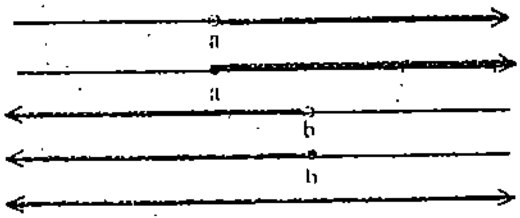
$]a, \infty[= \{x : a < x\}$ (open right ray)

$[a, \infty[= \{x : a \le x\}$ (closed right ray)

$]-\infty, b[= \{x : x < b\}$ (open left ray)

$]-\infty, b] = \{x : x \le b\}$ (closed left ray)

$]-\infty, \infty[= R$ (open interval)



As you can see easily, none of these sets are bounded. For instance, $]a, \infty[$ is bounded below, but is not bounded above. $]-\infty, b]$ is bounded above, but is not bounded below. Note that ∞ does not denote a real number. it merely indicates that an interval extends without limit.

We note further that if S is any interval (bounded or unbounded) and if c and d are two elements of S, then all numbers lying between c and d are also elements of S.

E 4) State whether the following are true or false.

a) $0 \in]1, 8]$, b) $-1 \in]-\infty, 2]$

c) $1 \in [1, 2]$ d) $5 \in [5, \infty[$

E 5) Represent the intervals in E 4) geometrically.

1.5 FUNCTIONS

Now let us move over to functions. Here we shall present some basic facts about functions which will help you refresh your knowledge. We shall look at various examples of functions and shall also define inverse functions. Let us start with the definition of a function.

1.5.1 Definition and Examples

Definition 4 If X and Y are two sets, a **function** f from X to Y, is a rule or a correspondence which connects every member of X to a unique member of Y. We write $f: X \to Y$ (read as "f is a function from X to Y"). X is called the domain and Y is called the co-domain of f. We shall denote by f(x) that unique element of Y which is associated to $x \in X$.

The following examples will help you in understanding this definition better.

Example 1 $f: N \to R$, defined by $f(x) = -x$. f is a function since the rule $f(x) = -x$ associates a unique member $(-x)$ of R to every member x of N. The domain here is N and the co-domain is R.

Example 2 The rule $f(x) = x/2$ does not define a function from $N \to Z$ as odd natural numbers like 1, 3, 5 … from N cannot be connected to any member of Z

Example 3 Every natural number can be written as a product of some prime numbers. Consider the rule f(x) = a prime factor of x, which connects elements of N. Here since $6 = 2 \times 3$, f(6) has two values: f(6) = 2 and f(6) = 3. This rule does not associate a unique number with 6 and hence does not give a function from N to N.

Thus, you see, to describe a function completely we have to specify the following three things:

a) the domain

b) the co-domain and

c) the rule which associates a unique member of the co-domain to each member of the domain.

The rule which defines a function need not always be in the form of a formula. But it should clearly specify (perhaps by actual listing) the correspondence between X and Y.

If $f: X \rightarrow Y$, then $y = f(x)$ is called the image of x under f or the f-image of x. The set of f-images of all members of X, i.e. $\{f(x) : x \in X\}$ is called the range of f and is denoted by f(X). It is easy to see that $f(X) \subseteq Y$.

Remark 3 a) Throughout this course we shall consider functions whose domain and co-domain are both subsets of R. Such functions are often called **real functions** or real-valued functions of a real variable. We shall, however, simply use the word 'function' to mean a real function.

b) The variable x used in describing a function is often called a dummy variable because it can be replaced by any other letter. Thus, for example, the rule $f(x) = -x, x \in N$ can as well be written in the form $f(t) = -t, t \in N$ or as $f(u) = -u, u \in N$. The variable x (or t or u) is also called an **independent variable**, and f(x) is **dependent** on this independent variable.

Graph of a function A convenient and useful method for studying a function is to study it through its graph. To draw the graph of a function $f : X \rightarrow Y$, we choose a system of coordinate axes in the plane. For each $x \in X$, the ordered pair (x, f(x)) determines a point in the plane (see Fig. 2). The set of all the points obtained by considering all possible values of x (remember that the domain of f is X) is the graph of f. The role that the graph of a function plays in the study of the function will become clear as we proceed further. In the meantime let us consider some more examples of functions and their graphs.

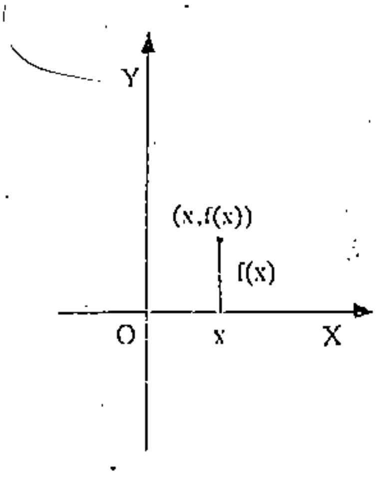


Fig. 2

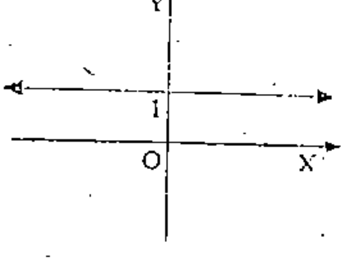


Fig. 3

1) A constant function: The simplest example of a function is a constant function. A constant function sends all the elements of the domain to just one element of the co-domain.

For example, let $f: R \rightarrow R$ be defined by $f(x) = 1$.

Alternatively, we may write

$f: x \rightarrow 1 \ \forall x \in R$

The graph of f is as shown in Fig. 3.

It is the line y = 1.

In general, the graph of a constant function $f : x \rightarrow c$ is a straight line which is parallel to the x-axis at a distance of $|c|$ units from it.

2 **The identity function**: Another simple but important example of a function is a function which sends every element of the domain to itself.

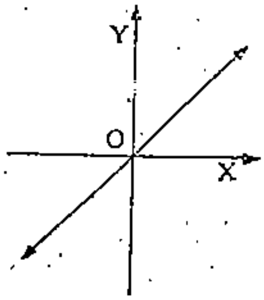


Fig. 4

Let X be any non-empty set, and let f be the function on X defined by setting $f(x) = x \ \forall x \in X$.

This function is known as the identity function on X and is denoted by i_X.

The graph of i_R the identity function on R, is shown in Fig. 4. It is the line y = x.

3 **Absolute value Function**: Another interesting function is the absolute value function (or modulus function) which can be defined by using the concept of the absolute value of a real number as:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0, \end{cases}$$

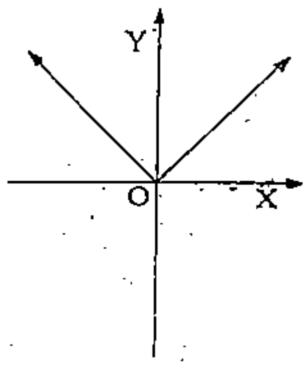
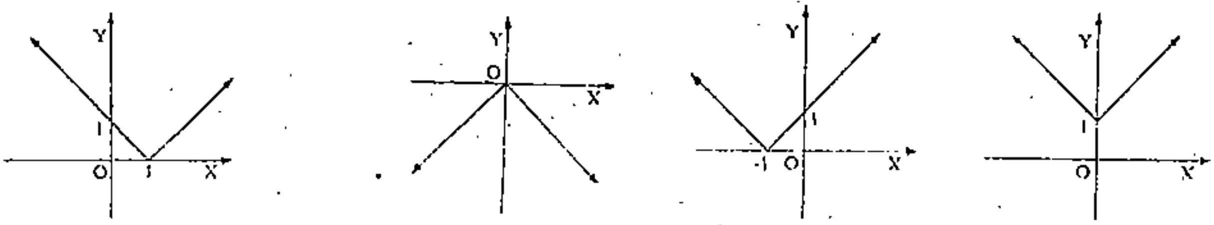


Fig. 5

The graph of this function is shown in Fig. 5. It consists of two rays, both starting at the origin and making angles $\pi/4$ and $3\pi/4$, respectively, with the positive direction of the x-axis.

E 6) Given below are the graphs of four functions depending on the notion of absolute value. The functions are $x \rightarrow -|x|$, $x \rightarrow |x| + 1$, $x \rightarrow |x + 1|$, $x \rightarrow |x - 1|$, though not necessarily in this order. (The domain in each case is R). Can you identify them?



(a) (b) (c) (d)

4 **The Exponential Function**: If a is a positive real number other than 1, we can define a function f as:

$f: R \rightarrow R$

$f(x) = a^x \qquad (a > 0, a \neq 1)$

This function is known as the **exponential function**. A special case of this function, where a=e, is often found useful. Fig. 6 shows the graph of the function $f : R \rightarrow R$ such that $f(x) = e^x$. This function is also called the **natural exponential function**. Its range is the set R^+ of positive real numbers.

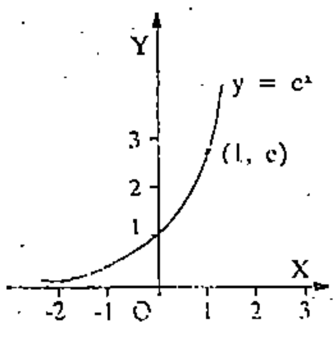


Fig. 6

5 **The Natural Logarithmic Function**: This function is defined on the set R^+ of positive real numbers $f: R^+ \rightarrow R$ such that $f(x) = \ln(x)$. The range of this function is R. Its graph is shown in Fig. 7.

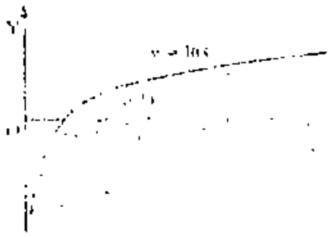


Fig. 7

6 **The Greatest Integer Function**: Take a real number x. Either it is an integer, say n (so that x = n) or it is not an integer. If it is not an integer, we can find (by the Archimedean property of real numbers) an integer n, such that $n < x < n + 1$. Therefore, for each real number x we can find an integer n such that $n \leq x < n + 1$. Further, for a given real number x, we can find only one such integer n. We say that n is the greatest integer not exceeding x, and denote it by [x]. For example, [3] = 3 and [3.5] = 3, [−3. 5] = − 4. Let us consider the function defined on R by setting f(x) = [x].

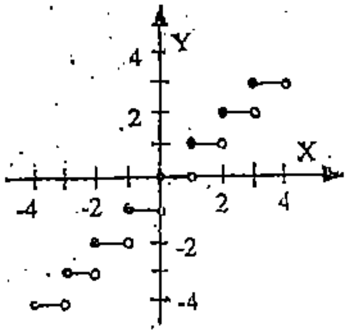


Fig. 8

This function is called the **greatest integer function.** The graph of the function is as shown in Fig. 8. (It resembles the steps of an infinite staircase).

Notice that the graph consists of infinitely many line segments of unit length, all parallel to the x - axis.

7 Other Functions The following are some important classes of functions.

a) **Polynomial Functions** $f(x) = a_0x^n + a_1x^{n-1} + \text{---------} + a_n$ where a_0, a_1, ------------, a_n are given real numbers (constants) and n is a positive integer.

b) **Rational Functions** $f(x) = g(x)/k(x)$, where g (x) and k (x) are polynomial functions of degree n and m. This is defined for all real x, for which $k(x) \neq 0$.

c) **Trigonometric or Circular Functions** $f(x) = \sin x$, $f(x) = \cos x$, $f(x) = \tan x$, $f(x) = \cot x$, $f(x) = \sec x$, $f(x) = \operatorname{cosec} x$.

d) **Hyperbolic Functions** $f(x) = \cosh x = \frac{(e^x + e^{-x})}{2}$, $f(x) = \sinh x = \frac{(e^x - e^{-x})}{2}$. We shall study these in detail in Unit 5.

1.5.2 Inverse Functions

In this sub-section we shall see what is meant by the inverse of a function. But before talking about the inverse, let us look at some special categories of functions. These special types of functions will then lead us to the definition of the inverse of a function.

One-one and Onto Functions

Consider the function $h : x \to x^2$, defined on the set R. Here $h(2) = h(-2) = 4$. 2 and -2 are distinct members of the domain R, but their h-images are the same. (Can you find some more numbers whose h-images are equal?) This may be expressed by saying that $\exists x, y$ such that $x \neq y$ but $h(x) = h(y)$.

Now, consider the function $g: x \to 2x + 3$

Here you will be able to see that if x_1 and x_2 are two distinct real numbers, then $g(x_1)$ and $g(x_2)$ are also distinct.

For, $x_1 \neq x_2 \Rightarrow 2x_1 \neq 2x_2 \Rightarrow 2x_1 + 3 \neq 2x_2 + 3 \Rightarrow g(x_1) \neq g(x_2)$

We have considered two functions here. While one of them, namely g, sends distinct members of the domain to distinct members of the co-domain, the other, namely h, does not always do so. We give a special name to functions like g above.

Definition 5 A function $f: X \to Y$ is said to be a **one-one function** (a (1 – 1) function or an injective function) if the images of distinct members of X are distinct members of Y.

Thus the function g above is one-one, whereas h is not one-one.

Remark 4 The condition "the images of distinct members of X are distinct members of Y" in the above definition can be replaced by either of the following equivalent conditions:

a) For every pair of members x, y of X, $x \neq y \Rightarrow f(x) \neq f(y)$

b) For every pair of members x,y of X, $f(x) = f(y) \Rightarrow x = y$.

We have observed earlier that for a function $f:X \to Y$,

$f(X) \subseteq Y$. This opens two possibilities:

i) $f(X) = Y$, or ii) $f(X) \subsetneq Y$, that is, f(X) is a proper subset of Y.

The function $h : x \to x^2 \ \forall x \in R$ falls in the second category. Since the square of any real number is always non-negative, $h(R) = R^+ \cup \{0\}$, the set of non-negative real numbers. Thus $h(R) \subsetneq R$.

On the other hand, the function $g : x \to 2x + 3$ belongs to the first category. Given any $y \in R$ (co-domain) if we take $x = (1/2)y - 3/2$, we find that $g(x) = y$. This shows that every member of the co-domain is a g-image of some member of the domain and thus, is in the range g(R). From this we get that $g(R) = R$. The following definition characterises this property of the function.

Definition 6 A function $f: X \to Y$ is said to be an **onto function** (or a surjective function) if every member of Y is the image of some member of X. If f is a function from X onto Y, we often write: $f : X \xrightarrow{\text{onto}} Y$ (or $f : X \to\!\!\to Y$).

Thus, h is not an onto function, whereas g is an onto function. Functions which are both one-one and onto are of special importance in mathematics. Let us see what makes them special.

Consider a function $f: X \to Y$ which is both one-one and onto. Since f is an onto function, each $y \in Y$ is the image of some $x \in X$. Also, since f is one-one, y cannot be the image of two distinct members of X. Thus, we find that to each $y \in Y$ there corresponds a unique $x \in X$ such that $f(x) = y$. Consequently, f sets up a one-to-one correspondence between the members of X and Y. It is this one-to-one correspondence between members of X and Y which makes a one-one and onto function so special, as we shall soon see.

Consider the function $f : N \to E$ defined by $f(x) = 2x$, where E is the set of even natural numbers. We can see that f is one-one as well as onto. In fact, to each $y \in E$ there exists $y/2 \in N$, such that $f(y/2) = y$. The correspondence $y \to y/2$ defines a function, say g, from E to N such that $g(y) = y/2$.

The function g so defined is called an inverse of f. Since, to each $y \in E$ there corresponds a unique $x \in N$ such that $f(x) = y$, only one such function g can be defined corresponding to a given function f. For this reason g is called the inverse of f.

As you will notice, the function g is also one-one and onto and therefore it will also have an inverse. You must have already guessed that the inverse of g is the function f.

From this discussion we have the following:

If f is a one-one and onto function from X to Y, then there exists a unique function $g : Y \to X$ such that for each $y \in Y$, $g(y) = x \Leftrightarrow y = f(x)$. The function g so defined is called the **inverse** of f. Further, if g is the inverse of f, then f is the inverse of g, and the two functions f and g are said to be the inverses of each other. The inverse of a function f is usually denoted by f^{-1}.

To find the inverse of a given function f, we proceed as follows:

Solve the equation $f(x) = y$ for x. The resulting expression for x (in terms of y) defines the inverse function.

Thus, if $f(x) = \frac{x^5}{5} + 2$, we solve $\frac{x^5}{5} + 2 = y$ for x.
This gives us $x = [5(y-2)]^{\frac{1}{5}}$. Hence f^{-1} is the function defined by $f^{-1}(y) = [5(y-2)]^{\frac{1}{5}}$

1.5.3 Graphs of Inverse Functions

There is an interesting relation between the graphs of a pair of inverse functions because of which, if the graph of one of them is known, the graph of the other can be obtained easily.

Let $f: X \to Y$ be a one-one and onto function, and let $g : Y \to X$ be the inverse of f. A point (p, q) lies on the graph of $f \Leftrightarrow q = f(p) \Leftrightarrow p = g(q) \Leftrightarrow$ (q, p) lies on the graph of g. Now the points (p, q) and (q, p) are reflections of each other with respect to (w.r.t.) the line $y = x$. Therefore, we can say that the graphs of f and g are reflections of each other w.r.t. the line $y = x$.

Therefore, it follows that, if the graph of one of the functions f and g is given, that of the other can be obtained by reflecting it w.r.t. the line $y = x$. As an illustration, the graphs of the functions $y = x^3$ and $y = x^{1/3}$ are given in Fig.9.

Do you agree that these two functions are inverses of each other? If the sheet of paper on which the graphs have been drawn is folded along the line $y = x$, the two graphs will exactly coincide.

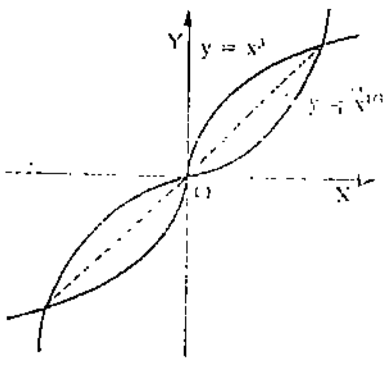


Fig. 9

E 7) Compare the graphs of $\ln x$ and e^x given in Figs. 6 and 7 and verify that they are inverses of each other.

If a given function is not one-one on its domain, we can choose a subset of the domain on which it is one-one, and then define its inverse function. For example, consider the function $f : x \to \sin x$.

Since we know that $\sin (x + 2\pi) = \sin x$, obviously this function is not one-one on **R**. But if we restrict it to the interval $[-\pi/2, \pi/2]$, we find that it is one-one. Thus if $f(x) = \sin x$ $\forall x \in [-\pi/2, \pi/2]$, then we can define
$f^{-1}(x) = \sin^{-1}(x) = y$ if $\sin y = x$.

Similarly, we can define $\cos^{-1}$ and $\tan^{-1}$ functions as inverse of cosine and tangent functions if we restrict the domains to $[0, \pi]$ and $]-\pi/2, \pi/2[$, respectively.

E 8) Which of the following functions are one-one?

a) $f: R \to R$ defined by $f(x) = |x|$

b) $f: R \to R$ defined by $f(x) = 3x - 1$

c) $f: R \to R$ defined by $f(x) = x^2$

d) $f: R \to R$ defined by $f(x) =$ [illegible]

E 9) Which of the following functions are onto?

a) $f: R \to R$ defined by $f(x) = 3x+7$

b) $f: R^+ \to R$ defined by $f(x) = \sqrt{x}$

c) $f: R \to R$ defined by $f(x) = x^2+1$

d) $f: X \to R$ defined by $f(x) = 1/x$,

where X stands for the set of non-zero real numbers.

E 10) Show that the function $f: X \to X$ such that $f(x) = \frac{x+1}{x-1}$, where X is the set of all real numbers except 1, is one-one and onto. Find its inverse.

E 11) Give one example of each of the following:

a) a one-one function which is not onto.

b) onto function which is not one-one.

c) a function which is neither one-one nor onto.

1.6 NEW FUNCTIONS FROM OLD

In this section we shall see how we can construct new functions from some given functions. This can be done by operating upon the given functions in a variety of ways. We give a few such ways here.

1.6.1 Operations on Functions

Scalar Multiple of a Function

Consider the function $f : x \to 3x^2+1 \quad \forall x \in R$. The function $g : x \to 2(3x^2+1) \quad \forall x \in R$ is such that $g(x) = 2f(x) \quad \forall x \in R$. We say that $g = 2f$, and that g is a scalar multiple of f by 2. In the above example there is nothing special about the number 2. We could have taken any real number to construct a new function from f. Also, there is nothing special about the particular function that we have considered. We could as well have taken any other function. This suggests the following definition: Let f be a function with domain D and let k be any real number. The scalar multiple of f by k is a function with domain D. It is denoted by kf and is defined by setting $(kf)(x) = kf(x)$.

Two special cases of the above definition are important.

i) Given any function f, if $k = 0$, the function kf turns out to be the zero function. That is, $0.f = 0$.

ii) If $k = -1$, the function kf is called the negative of f and is denoted simply by $-f$ instead of the clumsy $-1f$.

Absolute Value Function (or modulus function) of a given function

Let f be a function with domain D. The absolute value function of f, denoted by $|f|$ and read as mod f is defined by setting

$(|f|)(x) = |f(x)|$, for all $x \in D$.

Since $|f(x)| = f(x)$, if $f(x) \geq 0$, f and $|f|$ have the same graph for those values of x for which $f(x) \geq 0$.

Now let us consider those values of x for which $f(x) < 0$.

Here $|f(x)| = -f(x)$. Therefore, the graphs of f and $|f|$ are reflections of each other w.r.t. the x-axis for those values of x for which $f(x) < 0$.

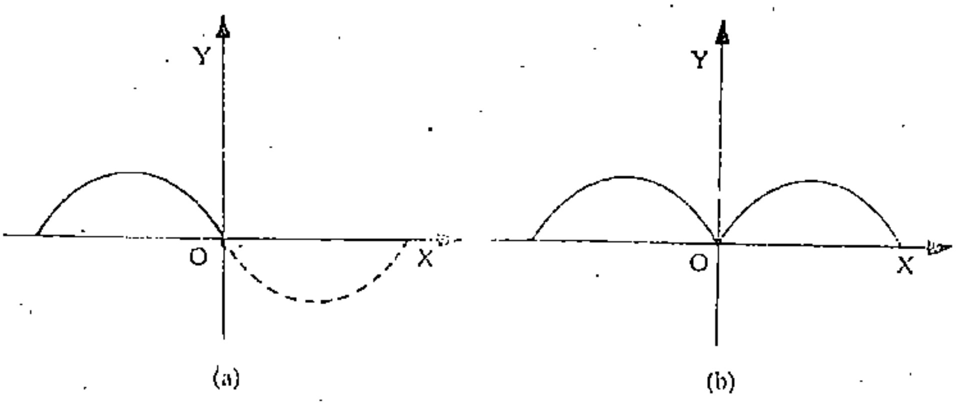


Fig. 10

As an example, consider the graph in Fig. 10(a). The portion of the graph below the x-axis (that is, the portion for which $f(x) < 0$) has been shown by a dotted line.

To draw the graph of $|f|$ we retain the undotted portion in Fig. 10(a) as it is, and replace the dotted portion by its reflection w.r.t. the x-axis (see Fig. 10(b)).

Sum, difference, Product and Quotient of two functions

If we are given two functions with a common domain, we can form several new functions by applying the four fundamental operations of addition, subtraction, multiplication and division on them.

i) Define a function s on D by setting

$s(x) = f(x) + g(x)$.

The function s is called the sum of the functions f and g, and is denoted by f + g. Thus,

$(f + g)(x) = f(x) + g(x)$

ii) Define a function d on D by setting

$d(x) = f(x) - g(x)$.

The function d is the function obtained by subtracting g from f, and is denoted by f−g. Thus, for all $x \in D$

$(f - g)(x) = f(x) - g(x)$.

iii) Define a function p on D by setting

$p(x) = f(x)\, g(x)$.

The function p, called the product of the function f and g, is denoted by fg. Thus, for all $x \in D$

$(fg)(x) = f(x)\, g(x)$

iv) Define a function q on D by setting $q(x) = f(x)/g(x)$, provided $g(x) \neq 0$ for any $x \in D$. The function q is called the quotient of f by g and is denoted by f/g. Thus,

$(f/g)(x) = f(x)/g(x) \quad (g(x) \neq 0$ for any $x \in D)$.

Remark 5 In case $g(x) = 0$ for some $x \in D$, we can consider the set, say D^*, of all those values of x for which $g(x) \neq 0$, and define f/g on D^* by setting $(f/g)(x) = f(x)/g(x) \ \forall x \in D^*$.

Example 4 Consider the functions $f : x \to x^2$ and $g : x \to x^3$. Then the functions f + g, f − g, fg are defined as

$(f + g)(x) = x^2 + x^3$,

$(f - g)(x) = x^2 - x^3$,

$(fg)(x) = x^5$

Now, $g(x) = 0 \Leftrightarrow x^3 = 0 \Leftrightarrow x = 0$. Therefore, in order to define the function f/g, we shall consider only non-zero values of x. If $x \neq 0$, $f(x)/g(x) = x^2/x^3 = 1/x$. Therefore f/g is the function

$f/g : x \to 1/x$, whenever $x \neq 0$.

All the operations defined on functions till now, were similar to the corresponding operations on real numbers. In the next subsection we are going to introduce an operation which has no parallel in R. Composite functions play a very important role in calculus. You will realise this as you read this course further.

1.6.2 Composite of Functions

We shall now describe a method of combining two functions which is somewhat different from the ones studied so far. Until now we have considered functions with the same domain. We shall now consider a pair of functions such that the co-domain of one is the domain of the other.

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. We define a function $h : X \to Z$ by setting $h(x) = g(f(x))$

To obtain h(x), we first take the f-image, f(x), of an element x of X. This $f(x) \in Y$, which is the domain of g. We then take the g-image of f(x), that is, g(f(x)), which is an element of Z. This scheme has been shown in Fig. 11.

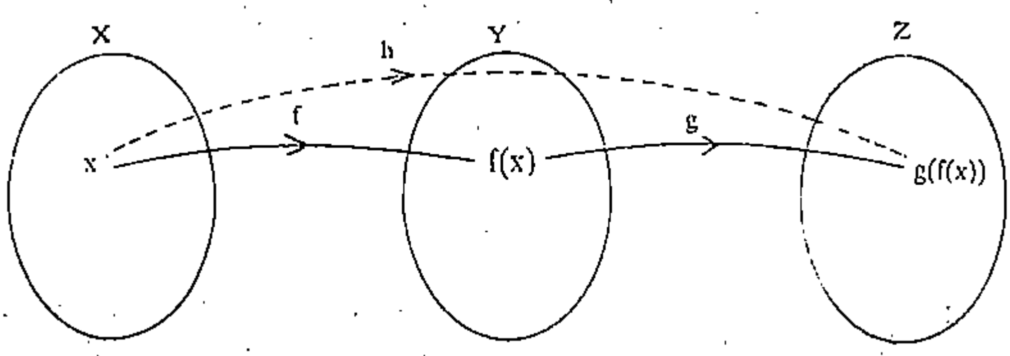


Fig. 11

The function h, defined above, is called the composite of f and g and is written as $g_\circ f$. Note the order. We first find the f-image and then its g-image. Try to distinguish it from $f_\circ g$, which will be defined only when Z is a subset of X.

Example 5 Consider the functions $f : x \to x^2 \ \forall x \in R$ and $g : x \to 8x + 1 \ \forall x \in R$. $g_\circ f$ is a function from R to itself, defined by $(g_\circ f)(x) = g(f(x)) = g(x^2) = 8x^2 + 1 \ \forall x \in R$. $f_\circ g$ is a function from R to itself defined by $(f_\circ g)(x) = f(g(x)) = f(8x + 1) = (8x+1)^2$. Thus $g_\circ f$ and $f_\circ g$ are both defined, but are different from each other.

The concept of composite functions is used not only to combine functions, but also to look upon a given function as made up of two simpler functions. For example, consider the function

$h : x \to \sin(3x + 7)$

We can think of it as the composite $(g_\circ f)$ of the functions $f : x \to 3x + 7 \ \forall x \in R$ and $g : u \to \sin u \ \forall u \in R$.

Now let us try to find the composites $f_\circ g$ and $g_\circ f$ of the functions:

$f : x \to 2x + 3 \ \forall x \in R$, and $g : x \to (1/2)x - 3/2 \ \forall x \in R$

Note that f and g are inverses of each other. Now $g_\circ f(x) = g(f(x)) = g(2x+3)$

$= \frac{1}{2}(2x + 3) - \frac{3}{2} = x$.

Similarly, $f_\circ g(x) = f(g(x)) = f(x/2 - 3/2) = 2(x/2 - 3/2) + 3 = x$. Thus, we see that $g_\circ f(x) = x$ and $f_\circ g(x) = x$ for all $x \in R$. Or, in other words, $g_\circ f$ and $f_\circ g$ are the identity function on R. What we have observed here is true for any two functions f and g which are inverses of each other. Thus, **if $f : X \to Y$ and $g : Y \to X$ are inverses of each other, then $g_\circ f$ and $f_\circ g$ are identity functions.** Since the domain of $g_\circ f$ is X and that of $f_\circ g$ is Y, we can write this as :

$g_\circ f = i_X, \ f_\circ g = i_Y$.

This fact is often used to test whether two given functions are inverses of each other.

1.7 TYPES OF FUNCTIONS

In this section we shall talk about various types of functions, namely, even, odd, increasing, decreasing and periodic functions. In each case we shall also try to explain the concept through graphs.

1.7.1 Even and Odd Functions

We shall first introduce two important classes of functions: even functions and odd functions. Consider the function f defined on R by setting

$f(x) = x^2 \ \forall x \in R$.

You will notice that $f(-x) = (-x)^2 = x^2 = f(x) \ \forall x \in R$

This is an example of an even function. Let's take a look at the graph (Fig 12) of this function. We find that the graph (a parabola) is symmetrical about the y-axis. If we fold the paper along the y-axis, we shall see that the parts of the graph on both sides of the y-axis completely coincide with each other. Such functions are called even functions. Thus, a function f, defined on R is even, if, for each $x \in R$, $f(-x) = f(x)$.

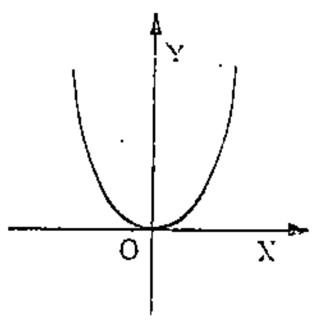
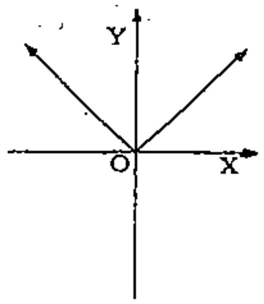


Fig. 12

The graph of an even function is symmetric with respect to the y-axis. We also note that if the graph of a function is symmetric with respect to the y-axis, the function must be an even function. Thus, if we are required to draw the graph of an even function, we can use this property to our advantage. We only need to draw that part of the graph which lies to the right of the y-axis and then just take its reflection w.r.t. the y-axis to obtain the part of the graph which lies to the left of the y-axis.

E E 12) Given below are two examples of even functions, alongwith their graphs. Try to convince yourself, by calculations as well as by looking at the graphs, that both the functions are, indeed, even functions.



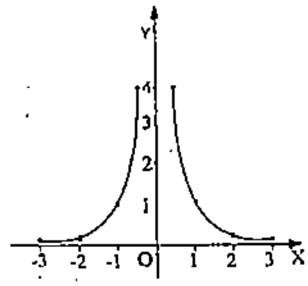
(a)

a) The absolute value function on R

$f : x \to |x|$

The graph of f is shown alongside.

b) The function g defined on the set of non-zero real numbers by setting $g(x) = 1/x^2$, $x \neq 0$.
The graph of g is shown alongside.



(b)

Now let us consider the function f defined by setting $f(x) = x^3 \;\forall x \in \mathbf{R}$. We observe that $f(-x) = (-x)^3 = -x^3 = -f(x) \;\forall x \in \mathbf{R}$. If we consider another function g given by $g(x) = \sin x$ we shall be able to note again that $g(-x) = \sin(-x) = -\sin x = -g(x)$.

The functions f and g above are similar in one respect: the image of $-x$ is the negative of the image of x. Such functions are called odd functions. Thus, a function f defined on **R** is said to be an odd function if $f(-x) = -f(x) \;\forall x \in \mathbf{R}$.

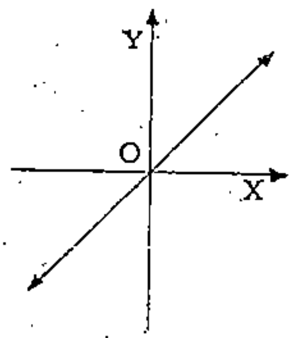
If $(x, f(x))$ is a point on the graph of an odd function f, then $(-x, -f(x))$ is also a point on it. This can be expressed by saying that the graph of an odd function is symmetric with respect to the origin. In other words, if you turn the graph of an odd function through 180° about the origin you will find that you get the original graph again. Conversely, if the graph of a function is symmetric with respect to the origin, the function must be an odd function. The above facts are often useful while handling odd functions.

E E 13) We are giving below two functions alongwith their graphs. By calculations as well as by looking at the graphs, find out whether each is even or odd.

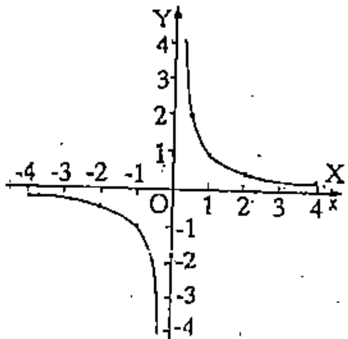
a) The identity function on **R**:

$f : x \to x$

b) The function g defined on the set of non-zero real numbers by setting $g(x) = 1/x$, $x \neq 0$.



(a)



(b)

While many of the functions that you will come across in this course will turn out to be either even or odd, there will be many more which will be neither even nor odd. Consider, for example, the function

$f : x \to (x+1)^2$

Here $f(-x) = (-x+1)^2 = x^2 - 2x + 1$. Is $f(x) = f(-x)\ \forall x \in R$?

The answer is 'no'. Therefore, f is not an even function. Is $f(x) = -f(-x)\ \forall x \in R$? Again, the answer is 'no'. Therefore f is not an odd function. The same conclusion could have been drawn by considering the graph of f which is given in Fig.13.

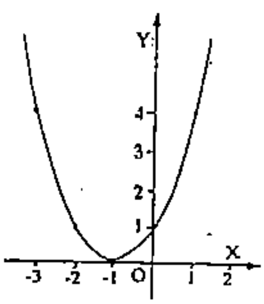


Fig. 13

You will observe that the graph is symmetric neither with respect to the y-axis, nor with respect to the origin.

Now there should be no difficulty in solving the exercise below.

E 14) Which of the following functions are even, which are odd, and which are neither even nor odd?

a) $x \to x^2+1\ \forall x \in R$

b) $x \to x^3-1,\ \forall x \in R$

c) $x \to \cos x,\ \forall x \in R$

d) $x \to x\,|x|\ \forall x \in R$

e) $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

1.7.2 Monotone Functions

In this sub-section we shall consider two types of functions:

i) Increasing and ii) Decreasing

Any function which conforms to any one of these types is called a monotone function. Does the profit of a company increase with production? Does the volume of gas decrease with increase in pressure? Problems like these require the use of increasing or decreasing functions. Now let us see what we mean by an increasing function. Consider the functions g and h defined by

$g(x) = x^3$ and $h(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$

Note that whenever $x_2 > x_1$, we get $x_2^3 > x_1^3$, that is, $g(x_2) > g(x_1)$.

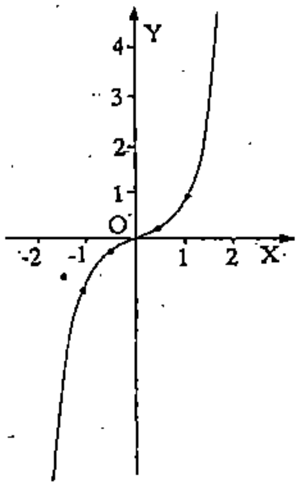


Fig. 14

In other words, as x increases, g(x) also increases. This fact can also be seen from the graph of g shown in Fig.14.

Let us find out how h(x) behaves as x increases. In this case we see that if $x_2 > x_1$, then $h(x_2) \ge h(x_1)$. (You can verify this by choosing any values for x_1 and x_2). Equivalently, we can say that h(x) increases (or does not decrease) as x increases. The same can be seen from the graph of h in Fig.15.

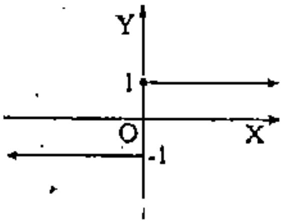


Fig.15

Functions like g and h above are called increasing or non-decreasing functions. Thus, a function f defined on a domain D is said to be increasing (or non-decreasing) if, for every pair of elements $x_1, x_2 \in D$, $x_2 > x_1 \Rightarrow f(x_2) \ge f(x_1)$. Further, we say that f is strictly increasing if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ (strict inequality).

Clearly, the function $g : x \to x^3$ discussed above, is a strictly increasing function; while h is not a strictly increasing function.

We shall now study another concept which is, in some sense, complementary to that of an increasing function.

Consider the function f_1 defined on R by setting

$$f_1(x) = \begin{cases} 1, & \text{if } x \le -1 \\ -x, & \text{if } -1 < x < 1 \\ -1, & \text{if } x \ge 1 \end{cases}$$

The graph of f_1 is as shown in Fig.16.

From the graph we can easily see that as x increases f_1 does not increase.

That is, $x_2 > x_1 \Rightarrow f_1(x_2) \le f_1(x_1)$ or $f_1(x_2) \not> f_1(x_1)$

Now consider the function $f_2 : x \to -x^3$ $(x \in R)$

The graph of f_2 is shown in Fig. 17.

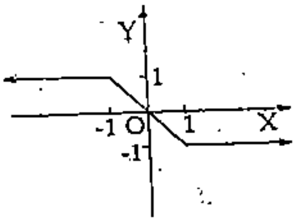


Fig.16

Since $x_2 > x_1 \Rightarrow x_2^3 > x_1^3 \Rightarrow -x_2^3 < -x_1^3 \Rightarrow f_2(x_2) < f_2(x_1)$, we find that as x increases, $f_2(x)$ decreases. Functions like f_1 and f_2 are called decreasing or non-increasing functions. The above two examples suggest the following definition:

A function f defined on a domain D is said to be **decreasing** (or **non-increasing**) if for every pair of elements x_1, x_2, $x_2 > x_1 \Rightarrow f(x_2) \le f(x_1)$. Further, f is said to be strictly decreasing if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

We have seen that, of the two decreasing functions f_1 and f_2, f_2 is strictly decreasing, while f_1 is not strictly decreasing. A function f defined on a domain D is said to be a **monotone function** if it is either increasing or decreasing on D.

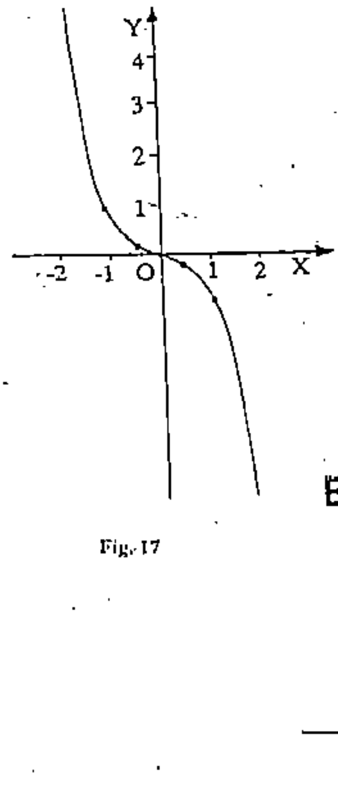


Fig.17

All the four functions (g, h, f_1, f_2) discussed above are monotone functions. The phrases 'monotonically increasing' and 'monotonically decreasing' are often used for 'increasing' and 'decreasing', respectively.

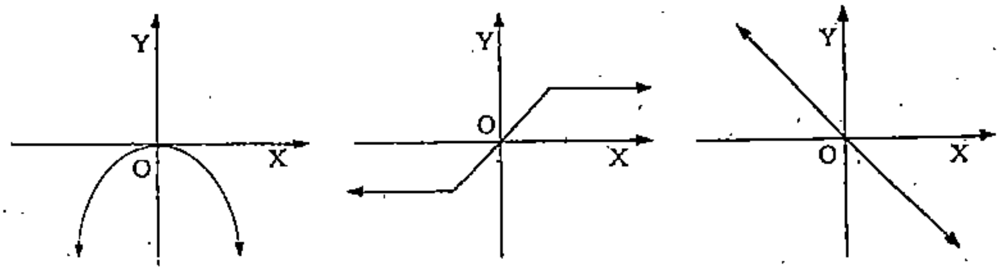
While many functions are monotone, there are many others which are not monotone. Consider, for example, the function

$f : x \to x^2$ $(x \in R)$.

You have seen the graph of f in Fig 12. This function is neither increasing nor decreasing.

If we find that a given function is not monotone, we can still determine some subsets of the domain on which the function is increasing or decreasing. For example, the function $f(x) = x^2$ is strictly decreasing in $]-\infty, 0]$ and is strictly increasing in $[0, \infty[$.

E E 15) Given below are the graphs of some functions. Classify them as non-decreasing, strictly decreasing, neither increasing nor decreasing:



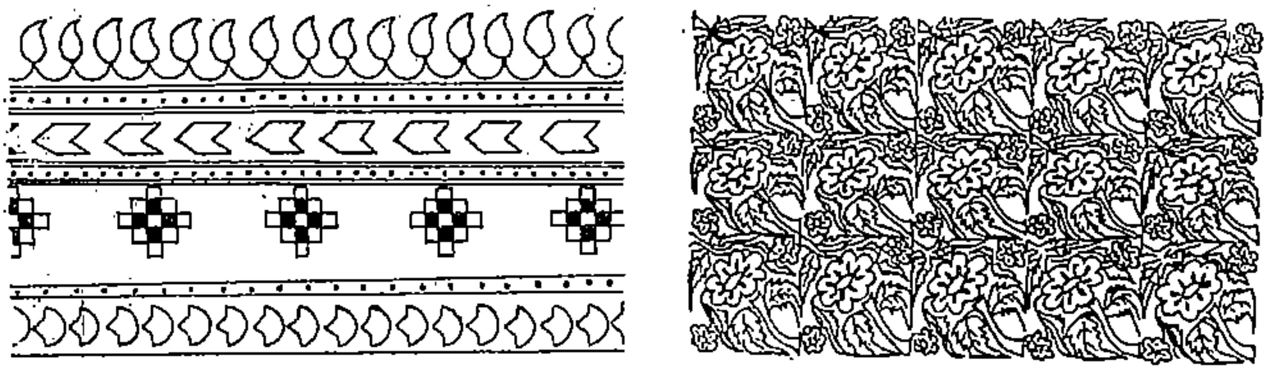
(a) (b) (c)

1.7.3 Periodic Functions

In this section we are going to tell you about yet another important class of functions, known as periodic functions.

Periodic functions occur very frequently in application of mathematics to various branches of science. Many phenomena in nature such as propagation of water waves, sound waves, light waves, electromagnetic waves etc. are periodic and we need periodic functions to describe them. Similarly, weather conditions and prices can also be described in terms of periodic functions.

Look at the following patterns:



(a) (b)

Fig. 18

You must have come across patterns similar to the ones shown in Fig.18 on the borders of sarees, wall papers etc. In each of these patterns a design keeps on repeating itself. A similar situation occurs in the graphs of periodic functions. Look at the graphs in Fig. 19.

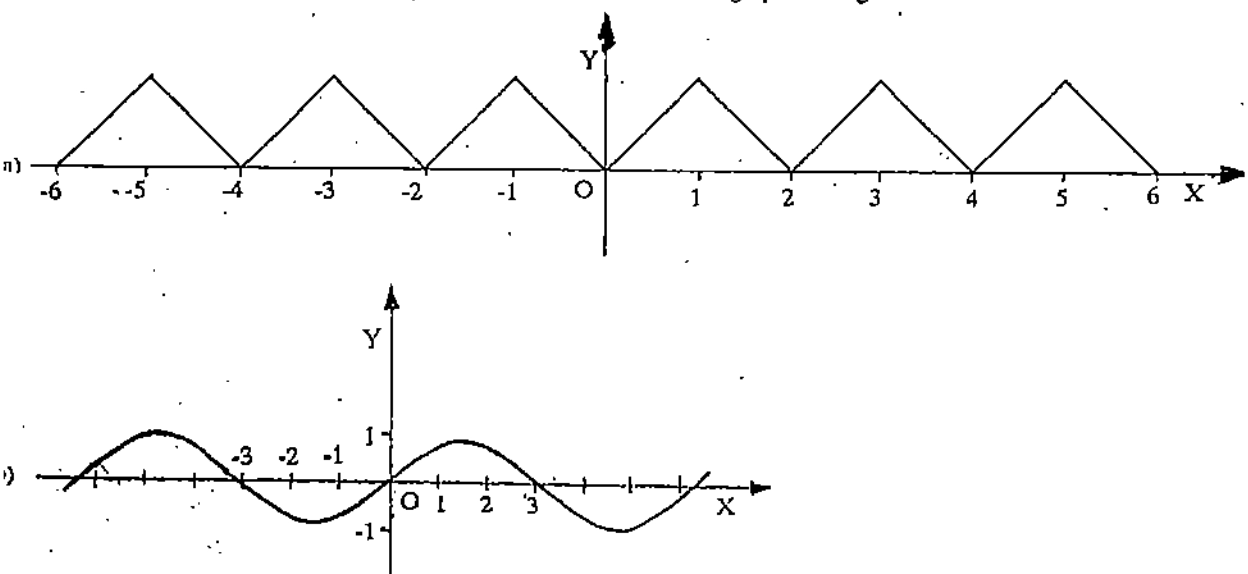


Fig. 19

n each of the figures shown above the graph consists of a certain pattern repeated infinitely nany times. Both these graphs represent periodic functions. To understand the situation, let is examine these graphs closely.

Consider the graph in Fig. 19(a). The portion of the graph lying between $x = -1$ and $x = 1$ is he graph of the function $x \to |x|$ on the domain $-1 \le x \le 1$.

This portion is being repeated both to the left as well as to the right, by translating (pushing) he graph through two units along the x-axis. That is to say, if x is any point of $[-1, 1]$, then he ordinates at x, $x \pm 2$, $x \pm 4$, $x \pm 6$, ... are all equal. The graph therefore represents the unction f defined by

$f(x) = |x|$, if $-1 \le x \le 1$ and $f(x + 2) = f(x)$.

The graph in Fig.19(b) is the graph of the sine function, $x \to \sin x$, $\forall x \in R$. You will notice hat the portion of the graph between 0 and 2π is repeated both to the right and to the left. You know already that $\sin(x + 2\pi) = \sin x$, $\forall x \in R$. We now give a precise meaning to the erm "a periodic function".

A function f defined on a domain D is said to be a periodic function if there exists a positive real number p such that $f(x + p) = f(x)$ for all $x \in D$. The number p is said to be a eriod of f.

f there exists a smallest positive p with the property described above it is called the period of f.

As you know, $\tan(x + n\pi) = \tan x$ $\forall n \in N$. This means that $n\pi$, $n \in N$ are all periods of the angent function. The smallest of these, that is π, is the period of the tangent function. See if you can do this exercise.

E E 16) a) What are the periods of the functions given in Fig. 19(a) and (b)?
b) Can you give one other period of each of these functions?

As another example of a periodic function, consider the function f defined on R by setting $f(x) = x - [x]$

Let us recall that $[x]$ stands for the greatest integer not exceeding x.

The graph of this function is as shown in Fig. 20.

From the graph (as also by calculation) we can easily see that

$f(x + n) = f(x)\ \forall x \in \mathbb{R}$, and for each positive integer n.

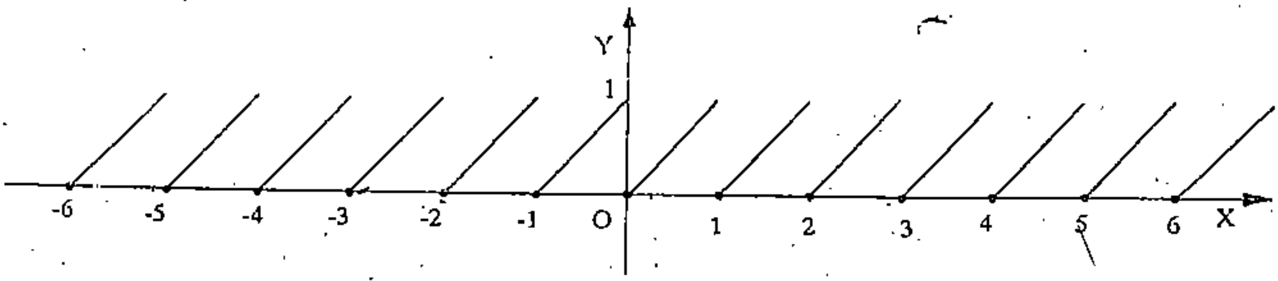


Fig. 20

The given function is therefore periodic, the numbers 1, 2, 3, 4 being all periods. The smallest of these, namely 1, is the period.

Thus the given function is periodic and has the period 1.

Remark 6 Monotonicity and periodicity are two properties of functions which cannot coexist. A monotone function can never be periodic, and a periodic function can never be monotone.

In general, it may not be easy to decide whether a given function is periodic or not. But sometimes it can be done in a straight forward manner. Suppose we want to find whether the function $f : x \to x^2\ \ \forall x \in \mathbb{R}$ is periodic or not. We start by assuming that it is periodic with period p

Then we must have $p > 0$ and $f(x + p) = f(x)\ \forall x$
$\Rightarrow (x + p)^2 = x^2 \quad \forall x$
$\Rightarrow 2xp + p^2 = 0 \quad \forall x$
$\Rightarrow p(2x + p) = 0 \quad \forall x$

Considering $x \neq -p/2$, we find that $2x + p \neq 0$. Thus, $p = 0$. This is a contradiction.

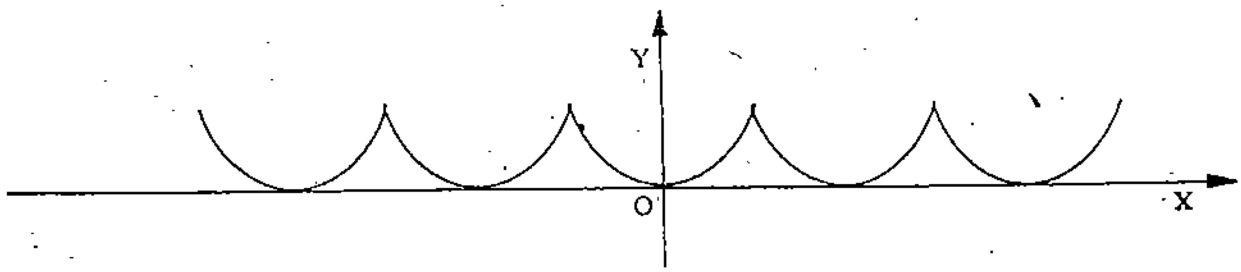
Therefore, there does not exist any positive number p such that $f(x+p) = f(x),\ \forall x \in \mathbb{R}$ and, consequently, f is not periodic.

E E 17) Examine whether the following functions are periodic or not. Write the periods of the periodic functions.

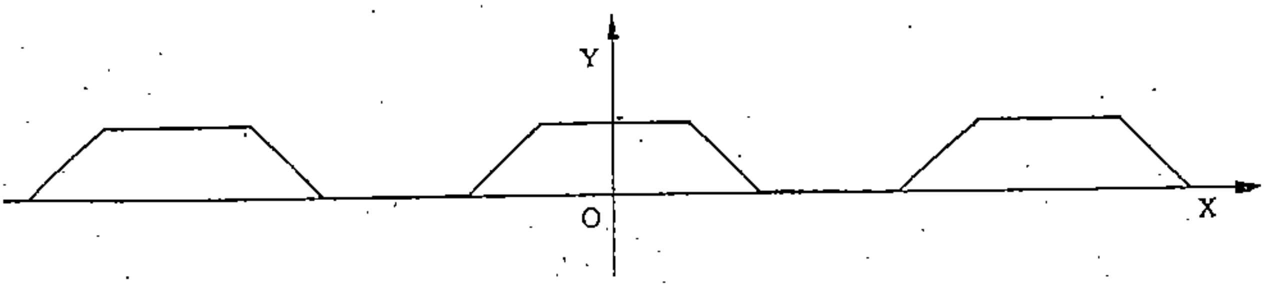
a) $x \to \cos x$ b) $x \to x + 2$
c) $x \to \sin 2x$ d) $x \to \tan 3x$
e) $x \to \cos(2x + 5)$ f) $x \to \sin x + \sin 2x$

E 18) The graphs of three functions are given below: classify the functions as periodic and non-periodic:

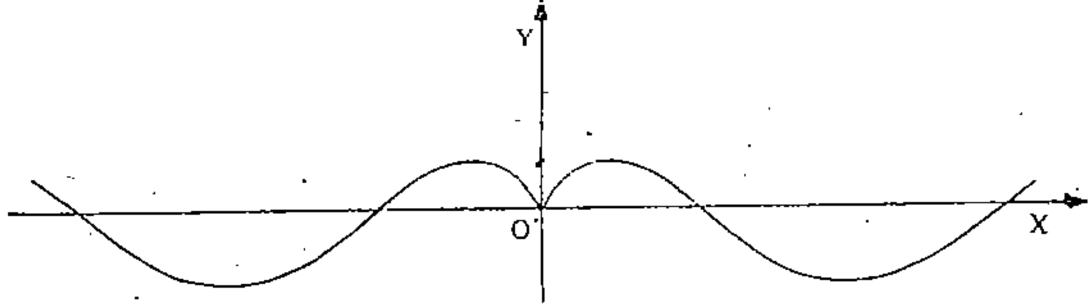
(a)



(b)



(c)



E 19) Is the sum of two periodic functions also periodic? Give reasons for your answer.

We end with summarising what we have discussed in this unit.

1.8 SUMMARY

In this unit we have

1 briefly revised the basic properties of real numbers.

2 defined the absolute value of a real number x as

$|x| = x$ if $x \geq 0$

$= -x$ if $x < 0$

3 discussed various types of intervals in **R**

Open : $]a,b[= \{x \in \mathbf{R} : a < x < b\}$

closed: $[a,b] = \{x \in \mathbf{R} : a \leq x \leq b\}$

semi-open: $]a,b] = \{x \in \mathbf{R} : a < x \leq b\}$

or $[a,b[= \{x \in \mathbf{R} : a \leq x < b\}$,

where $a, b \in \mathbf{R}$,

4 defined a function and discussed various types of functions along with their graphs: one-one, onto, even, odd, monotone, periodic.

5 defined composite of functions and discussed the existence of the inverse of a function.

1.9 SOLUTIONS AND ANSWERS

E 1) a) The set $\{1, 2, 3, \ldots\}$ has a lower bound, e.g., 0.

b) The set $\{\ldots\ldots\ldots, -3, -2, -1, 0, 1, 2, \ldots\ldots\ldots\}$ does not have a lower bound.

c) The g.l.b. of the set $S = \{1, 1/2, 1/3, \ldots\ldots, 1/n, \ldots\}$ is 0, and $0 \notin S$.

d) $\{x : x \in R$ and $1 \leq x \leq 2\}$ is a bounded set as it is bounded above by 2 and below by 1.

E 2) a) A real number p is positive if $p > 0$. Hence 0 is a lower bound for the set P of positive real numbers. Thus the set P is bounded below. Its infimum is 0.

b) A real number r is the infimum of a set $S \subset \mathbf{R}$ if and only if the following conditions are satisfied:

i) $r \leq x$ for all $x \in S$.

ii) For each $\varepsilon > 0$, there is $y \in S$ such that $y < r + \varepsilon$.

The set P in a) above has infimum 0, since

i) $0 < p$ for all $p \in P$ and

ii) for each $\varepsilon > 0$ there is $\varepsilon/2 \in P$ such that $\varepsilon/2 < 0 + \varepsilon = \varepsilon$

E3) a) $|x| = \max\{x, -x\}$. Hence $x = 0 \Rightarrow |x| = 0$ and

$|x| = 0 \Rightarrow \max.\{x, -x\} = 0 \Rightarrow x = 0$.

b) There are three cases 1) $x \geq 0, y \geq 0$

2) x and y have opposite signs

3) $x \leq 0, y \leq 0$.

We take 2). Suppose $x > 0, y < 0$, then,

$|x| = \max\{x, -x\} = x$, $|y| = \max\{y, -y\} = -y$.

$xy < 0 \Rightarrow |xy| = -xy = x \times (-y) = |x||y|$.

1) and 3) can be proved similarly.

c) If $x > 0$, $|x| = x$ and $|1/x| = 1/x = 1/|x|$

If $x < 0$, $|x| = -x$ and $|1/x| = -1/x = 1/|x|$

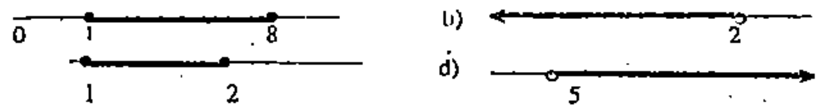
d) $|x - y| = |x + (-y)| \le |x| + |-y| = |x| + |y|$

e) $|x + y + z| = |(x + y) + z| \le |x + y| + |z| \le |x| + |y| + |z|$

f) $|xyz| = |xy||z| = |x||y||z|$

E 4) a) False b) True c) True d) False

E 5) a) b) c) d)



E 6) a) $x \to |x - 1|$ b) $x \to -|x|$

c) $x \to |x + 1|$ d) $x \to |x| + 1$

E 8) b) is one-one

E 9) a) is onto

E 10) $f(x) = \frac{x+1}{x-1} = 1 + \frac{2}{x-1}$

$f(x) = f(y) \Rightarrow 1 + \frac{2}{x-1} = 1 + \frac{2}{y-1}$

$$\Rightarrow \frac{2}{x-1} = \frac{2}{y-1} \Rightarrow x - 1 = y - 1 \Rightarrow x = y$$

Hence f is one-one.

If $y \in X$, put $x = \frac{y+1}{y-1}$. Then $x \in X$ and $y = f(x)$. Hence, f is onto.

$f^{-1}(x) = \frac{x+1}{x-1}$

E 11) a) $f : R^+ \longrightarrow R : f(x) = \sqrt{x}$

b) $f : R \longrightarrow R^+ \cup \{0\} : f(x) = x^2$

c) $f : R \longrightarrow R : f(x) = x^2$

E 12) a) $f(x) = |x| \Rightarrow f(-x) = |-x| = |x| = f(x)$. Hence, f is even.

b) $g(x) = 1/x^2 \Rightarrow g(-x) = 1/(-x)^2 = g(x)$. Hence, g is even.

E 13) a) $f(x) = x \Rightarrow f(-x) = -x = -f(x)$. Hence, f is odd.

b) $g(x) = 1/x \Rightarrow g(-x) = -1/x = -g(x)$. Hence, g is odd.

E 14) a), c) and e) are even

d) is odd

b) is neither even nor odd

E 15) a) neither increasing, nor decreasing

b) non-decreasing c) strictly decreasing

E 16) The period of the function in Fig.19 a) is 2. Other periods are 4, 6, 8, ...

The period of the function in Fig.19b) is 2π. Other periods are 4π, 6π, ...

E 17) a) Periodic with period 2π

Since $\cos(x + 2\pi) = \cos x$ for all x.

b) not periodic

c) Periodic with period π.

d) Periodic with period $\pi/3$.

e) Periodic with period π.

f) Periodic with period 2π.

E 18) a) and b) are periodic, c) is not.

E 19) No. For example, $x - [x]$ and $|\sin x|$ are periodic, but their sum is not.

UNIT 2 LIMITS AND CONTINUITY

Structure

2.1 INTRODUCTION

The last unit has helped you in recalling some fundamentals that will be needed in this course. We will now begin the study of calculus, starting with the concept of 'limit'. As you read the later units, you will realise that the seeds of calculus were sown as early as the third century B.C. But it was only in the nineteenth century that a rigorous definition of a limit was given by Weierstrass. Before him, Newton, d' Alembert and Cauchy had a clear idea about limits, but none of them had given a formal and precise definition. They had depended, more or less, on intuition or geometry.

The introduction of limits revolutionised the study of calculus. The cumbersome proofs which were used by the Greek mathematicians have given way to neat, simpler ones.

You may already have an intuitive idea of limits. In Sec.2 of this unit, we shall give you a precise definition of this concept. This will lead to the study of continuous functions in Sec.3. Most of the functions that you will come across in this course will be continuous. We shall also give you some examples of discontinuous functions.

Objectives

After reading this unit you should be able to:

- calculate the limits of functions whenever they exist,
- identify points of continuity and discontinuity of a function.

2.2 LIMITS

In this section we will introduce you to the notion of 'limit'. We start with considering a situation which a lot of us are familiar with, such as train travel. Suppose we are travelling from Delhi to Agra by a train which will reach Agra at 10.00 a.m. As the time gets closer and closer to 10.00 a.m., the distance of the train from Agra gets closer and closer to zero (assuming that the train is running on time!). Here, if we consider time as our independent variable, denoted by t and distance as a function of time, say f(t), then we see that f(t) approaches zero as t approaches 10. In this case we say that the limit of f(t) is zero as t tends to 10.

Now consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$. Let us consider Tables 1(a) and 1(b) in which we give the values of f(x) as x takes values nearer and nearer to 1.

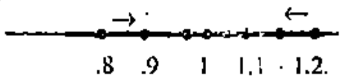
In Table 1(a) we see values of x which are greater than 1. We can also express this by saying that x approaches 1 from the right. Similarly, we can say that x approaches 1 from the left in Table 1(b).

Table 1(a)

x	1.2	1.1	1.01	1.001
f(x)	2.44	2.42	2.02	2.002

Table 1(b)

x	0.8	0.9	0.99	0.999
f(x)	1.64	1.81	1.9801	1.9989



We find that, as x gets closer and closer to 1, f(x) gets closer and closer to 2. Alternatively, we express this by saying that as x approaches 1 (or tends to 1), the limit of f(x) is 2. Let us now give a precise meaning of 'limit'.

Definition 1 Let f be a function defined at all points near p (except possibly at p). Let L be a real number. We say that f approaches the limit L as x approaches p if, for each real number $\varepsilon > 0$, we can find a real number $\delta > 0$ such that

$0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

This definition of limit was first stated by Karl Weierstrass, around 1850.

As you know from Unit 1, $|x - p| < \delta$ means that $x \in\ ]p - \delta, p + \delta[$ and $0 < |x - p|$ means that $x \neq p$. That is, $0 < |x - p| < \delta$ means that x can take any value lying between $p - \delta$ and $p + \delta$ except p.

(ε epsilon) and δ (delta) are Greek letters used to denote real numbers.

The limit L is denoted by $\lim_{x \to p} f(x)$. We also write $f(x) \to L$ as $x \to p$..

'$\to$' denotes 'tends to'

Note that, in the above definition, we take any real number $\varepsilon > 0$ and then choose some $\delta > 0$, so that $L - \varepsilon < f(x) < L + \varepsilon$, whenever $|x - p| < \delta$, that is, $p - \delta < x < p + \delta$.

The $\varepsilon - \delta$ definition does not give us the value of L. It just helps us check whether a given number L is the limit of f(x).

In Unit 1 we have also mentioned that $|x - p|$ can be thought of as the distance between x and p. In the light of this the definition of the limit of a function can also be interpreted as:

Given $\varepsilon > 0$, we can choose $\delta > 0$ such that if we choose x whose distance from p is less than δ, then the distance of its image from L must be less than ε. The pictures in Fig.1 may help you absorb the definition.

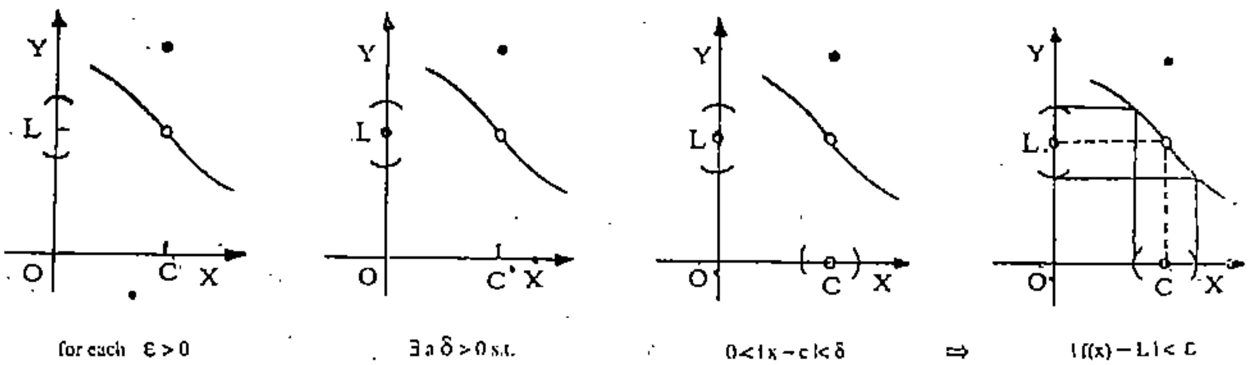


Fig.1

Remember, the number ε is given first and the number δ is to be produced.

An important point to note here is that while taking the limit of f(x) as $x \to p$, we are concerned only with the values of f(x) as x takes values closer and closer to p, but not when $x = p$. For example, consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$. This function is not defined for $x = 1$, but is defined for all other $x \in R$. However, we can still talk about its limit as $x \to 1$. This is because for taking the limit we will have to look at the values of f(x) as x tends to 1, but not when $x = 1$.

Now let us take the following examples:

Example 1 Consider the function $f : R \to R$ defined by $f(x) = x^3$. How can we find $\lim_{x \to 0} f(x)$?

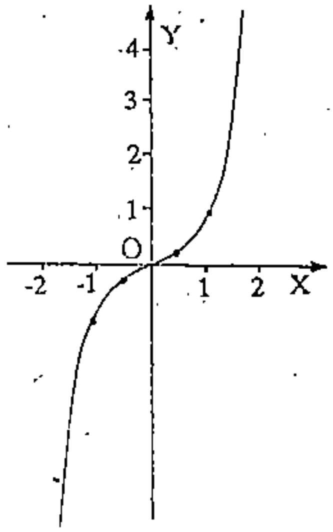


Fig.2

Look at the graph of f in Fig.2. You will see that when x is small, x^3 is also small. As x comes closer and closer to 0, x^3 also comes closer and closer to zero. It is reasonable to expect that $\lim f(x) = 0$ as $x \to 0$.

Let us prove that this is what happens. Take any real number $\varepsilon > 0$. Then, $| f(x) - 0 | < \varepsilon \Leftrightarrow | x^3 | < \varepsilon \Leftrightarrow | x | < \varepsilon^{1/3}$. Therefore, if we choose $\delta = \varepsilon^{1/3}$ we get $| f(x) - 0 | < \varepsilon$ whenever $0 < | x - 0 | < \delta$. This gives us $\lim\limits_{x \to 0} f(x) = 0$.

A useful general rule to prove $\lim\limits_{x \to a} f(x) = L$ is to write down $f(x) - L$ and then express it in terms of $(x - a)$ as much as possible.

Let us now see how to use this rule to calculate the limit in the following examples.

Example 2 Let us calculate $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$

We know that division by zero is not defined. Thus, the function $f(x) = \frac{x^2 - 1}{x - 1}$ is not defined at $x = 1$. But, as we have mentioned earlier, when we calculate the limit as x approaches 1, we do not take the value of the function at $x = 1$. Now, to obtain

$\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$, we first note that $x^2 - 1 = (x - 1)(x + 1)$, so that, $\frac{x^2 - 1}{x - 1} = x + 1$ for $x \neq 1$.

Therefore $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim\limits_{x \to 1} (x + 1)$

As x approaches 1, we can intuitively see that this limit approaches 2. To prove that the limit is 2, we first write $f(x) - L = x + 1 - 2 = x - 1$, which is itself in the form $x - a$, since $a = 1$ in this case. Let us take any number $\varepsilon > 0$. Now,

$|(x + 1) - 2| < \varepsilon \Leftrightarrow | x - 1 | < \varepsilon$

Thus, if we choose $\delta = \varepsilon$, in our definition of limit, we see that

$| x - 1 | < \delta = \varepsilon \Rightarrow | f(x) - L | = | x - 1 | < \varepsilon$. This shows that $\lim\limits_{x \to 1} (x + 1) = 2$. Hence,

$\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$.

Example 3 Let us prove that $\lim\limits_{x \to 3} (x^2 + 4) = 13$.

That is, we shall prove that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $| x^2 + 4 - 13 | < \varepsilon$ whenever $| x - 3 | < \delta$.

Here, $f(x) - L = (x^2 + 4) - 13 = x^2 - 9$, and $x - a = x - 3$.

Now we write $|x^2 - 9|$ in terms of $|x - 3|$:

$|x^2 - 9| = |x + 3|\,|x - 3|$

Thus, apart from $|x - 3|$, we have a factor, namely $|x + 3|$. To decide the limits of $|x + 3|$, let us put a restriction on δ. Remember, we have to choose δ. So let us say we choose a $\delta \le 1$. What does this imply?

$|x - 3| < \delta \Rightarrow |x - 3| < 1 \Rightarrow 3 - 1 < x < 3 + 1$

$\Rightarrow 2 < x < 4 \Rightarrow 5 < x + 3 < 7.$ Recall Sec.4, Unit 1.

Thus, we have $|x^2 - 9| < 7|x - 3|$. But our aim is to prove $|x^2 - 9| < \varepsilon$.

For this we shall try to make $7|x - 3| < \varepsilon$. Now when will this be true? It will be true when $|x - 3| < \varepsilon/7$. So this $\varepsilon/7$ is the value of δ we were looking for. But we have already chosen $\delta \le 1$. This means that given $\varepsilon > 0$, the δ we choose should satisfy

$\delta \le 1$ and also $\delta \le \varepsilon/7$.

In other words, $\delta = \min\{1, \varepsilon/7\}$, should serve our purpose. Let us verify this:

$|x - 3| < \delta \Rightarrow |x - 3| < 1$ and $|x - 3| < \varepsilon/7 \Rightarrow |x^2 - 9| = |x + 3|\,|x - 3| < 7 \cdot \varepsilon/7 = \varepsilon.$

Remark 1: If f is a constant function on R, that is, if $f(x) = k \; \forall x \in R$, where k is some fixed real number, then $\lim_{x \to p} f(x) = k$.

Now please try the following exercises.

E 1) Show that

a) $\lim_{x \to 1} \frac{1}{x} = 1$

b) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3.$

Before we go further, let us ask, 'Can a function f(x) tend to two different limits as x tends to p'?

The answer is NO, as you can see from the following :

Theorem 1 If $\lim_{x \to p} f(x) = L$ and $\lim_{x \to p} f(x) = M$, then $L = M$.

Proof : Suppose $L \neq M$, then $|L - M| > 0$. Since $\lim_{x \to p} f(x) = L$, if we take $\varepsilon = \frac{|L - M|}{2}$ then $\exists\, \delta_1 > 0$ such that

$|x - p| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$

Similarly, since $\lim_{x \to p} f(x) = M$, $\exists\, \delta_2 > 0$ such that

$|x - p| < \delta_2 \Rightarrow |f(x) - M| < \varepsilon$.

If we choose $\delta = \min\{\delta_1, \delta_2\}$, then $|x - p| < \delta$ will mean that $|x-p| < \delta_1$ and $|x - p| < \delta_2$. In this case we will have both $|f(x) - L| < \varepsilon$, as well as, $|f(x) - M| < \varepsilon$.

$|a + b| \leq |a| + |b|$

So that $|L - M| = |L - f(x) + f(x) - M| \leq |f(x) - L| + |f(x) - M| < \varepsilon + \varepsilon$
$= 2\varepsilon = |L - M|$.

That is, we get $|L - M| < |L - M|$, which is a contradiction. Therefore, our supposition is wrong. Hence $L = M$.

We now state and prove a theorem whose usefulness will be clear to you in Unit 4.

Theorem 2 Let f, g and h be functions defined on an interval I containing a, except possibly at a. Suppose

i) $f(x) \leq g(x) \leq h(x)\ \forall x \in I \setminus \{a\}$

ii) $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

Then $\lim_{x \to a} g(x)$ exists and is equal to L.

Proof: By the definition of limit, given $\varepsilon > 0$, $\exists\, \delta_1 > 0$ and $\delta_2 > 0$ such that

$|f(x) - L| < \varepsilon$ for $0 < |x - a| < \delta_1$ and

$|h(x) - L| < \varepsilon$ for $0 < |x - a| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then,

$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$

$\Rightarrow L - \varepsilon \leq f(x) \leq L + \varepsilon$, and

$L - \varepsilon \leq h(x) \leq L + \varepsilon$

We also have $f(x) \leq g(x) \leq h(x)\ \forall x \in I \setminus \{a\}$

Thus, we get $0 < |x - a| < \delta \Rightarrow L - \varepsilon \leq f(x) \leq g(x) \leq h(x) \leq L + \varepsilon$

In other words, $0 < |x - a| < \delta \Rightarrow |g(x) - L| < \varepsilon$

Therefore $\lim_{x \to a} g(x) = L$.

Theorem 2 is also called the sandwich theorem (or the squeeze theorem), because g is being sandwiched between f and h. Let us see how this theorem can be used.

Example 4 Given that $|f(x) - 1| \leq 3(x + 1)^2\ \forall x \in \mathbf{R}$, can we calculate $\lim_{x \to -1} f(x)$?

We know that $-3(x + 1)^2 \leq f(x) - 1 \leq 3(x + 1)^2\ \forall x$. This means that

$-3(x + 1)^2 + 1 \leq f(x) \leq 3(x + 1)^2 + 1\ \forall x$. Using the sandwich theorem and the fact that $\lim_{x \to -1}[-3(x + 1)^2 + 1] = 1 = \lim_{x \to -1}[3(x + 1)^2 + 1]$, we get $\lim_{x \to -1} f(x) = 1$.

In the next section we will look at the limits of the sum, product and quotient of functions.

2.2.1 Algebra of limits

Now that you are familiar with limits. Let us state some basic properties of limits. (Their proofs are beyond the scope of this course.)

Theorem 3 Let f and g be two functions such that

$\lim_{x \to p} f(x)$ and $\lim_{x \to p} g(x)$ exist. Then

i) $\lim_{x \to p} [f(x) + g(x)] = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$ — Sum rule

ii) $\lim_{x \to p} [f(x)\, g(x)] = \left[\lim_{x \to p} f(x)\right]\left[\lim_{x \to p} g(x)\right]$ — Product rule

iii) $\lim_{x \to p} \frac{1}{g(x)} = \frac{1}{\lim_{x \to p} g(x)}$, provided $\lim_{x \to p} g(x) \neq 0$ — Reciprocal rule

We can easily prove two more rules in addition to the three rules given in Theorem 3. These are:

iv) $\lim_{x \to p} k = k$ Constant function rule

v) $\lim_{x \to p} x = p$ Identity function rule

We shall only indicate the method of proving iv) and v):

iv) Here $| f(x) - L | = | k - k | = 0 < \varepsilon$, whatever be the value of δ.

v) $| f(x) - L | = | x - p | < \varepsilon$ whenever $| x - p | < \delta$, if we choose $\delta = \varepsilon$.

Using the properties that we have just stated, we will calculate the limit in the following example.

Example 5 Let us evaluate $\lim_{x \to 2} \frac{3x^2 + 4x}{2x + 1}$

Now $\lim_{x \to 2} 2x + 1 = \lim_{x \to 2} 2x + \lim_{x \to 2} 1$ by using i)

$= \lim_{x \to 2} 2 \lim_{x \to 2} x + \lim_{x \to 2} 1$ by using ii)

$= 2 \times 2 + 1 = 5 \neq 0$ by using iv) and v)

$\therefore$ We can use (iii) of Theorem 3. Then the required limit is

$$\frac{\lim_{x \to 2} (3x^2 + 4x)}{\lim_{x \to 2} (2x + 1)} = \frac{\lim_{x \to 2} 3x^2 + \lim_{x \to 2} 4x}{\lim_{x \to 2} 2x + \lim_{x \to 2} 1} \quad \text{by using i)}$$

$$= \frac{\lim_{x \to 2} 3 \lim_{x \to 2} x \lim_{x \to 2} x + \lim_{x \to 2} 4 \lim_{x \to 2} x}{\lim_{x \to 2} 2 \lim_{x \to 2} x + \lim_{x \to 2} 1} \quad \text{by using ii)}$$

$$= \frac{3 \times 2 \times 2 + 4 \times 2}{2 \times 2 + 1} = \frac{20}{5} = 4$$

You can similarly calculate the limits in the following exercises.

E E 2) Show that $\lim_{x \to 1} \frac{3}{x} = 3$

E E 3) Calculate $\lim_{x \to 1} 2x + 5 \left(\frac{x^2}{1 + x^2} \right)$

2.2.2 Limits as $x \to \infty$ (or $-\infty$)

Take a look at the graph of the function $f(x) = \frac{1}{x}$, $x > 0$ in Fig.3. This is a decreasing function of x. In fact, we see from Fig. 3 that f(x) comes closer and closer to zero as x gets larger and larger. This situation is similar to the one where we have a function g(x) getting closer and closer to a value L as x comes nearer and nearer to some number p, that is when $\lim_{x \to p} g(x) = L$.

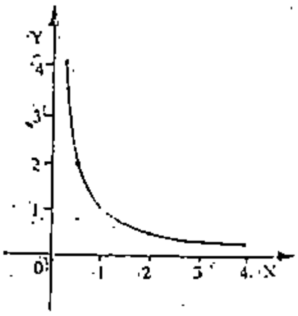


Fig. 3

The only difference is that in the case of f(x), x is not approaching any finite value, and is just becoming larger and larger. We express this by saying that $f(x) \to 0$ as $x \to \infty$, or $\lim_{x \to \infty} f(x) = 0$. Note that, ∞ is not a real number. We write $x \to \infty$ merely to indicate that x becomes larger and larger.

We now formalise this discussion in the following definition.

Definition 2 A function f is said to tend to a limit L as x tends to ∞ if, for each $\varepsilon > 0$ it is possible to choose $K > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > K$.

In this case, as x gets larger and larger, f(x) gets nearer and nearer to L. We now give another example of this situation.

Example 6 Let f be defined by setting $f(x) = 1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Here f is defined for all real values of x other than zero. Let us substitute larger and larger values of x in $f(x) = 1/x^2$ and see what happens (see Table 2).

Table 2

x	100	1000	100,000
$f(x) = 1/x^2$	.0001	.000001	.0000000001

We see that as x becomes larger and larger, f(x) comes closer and closer to zero. Now, let us choose any $\varepsilon > 0$. If $x > 1/\sqrt{\varepsilon}$, then $1/x^2 < \varepsilon$. Therefore, by choosing $K = 1/\sqrt{\varepsilon}$, we find that $x > K \Rightarrow |f(x)| < \varepsilon$. Thus, $\lim_{x \to \infty} f(x) = 0$.

Fig.4 gives us a graphic idea of how this function behaves as $x \to \infty$.

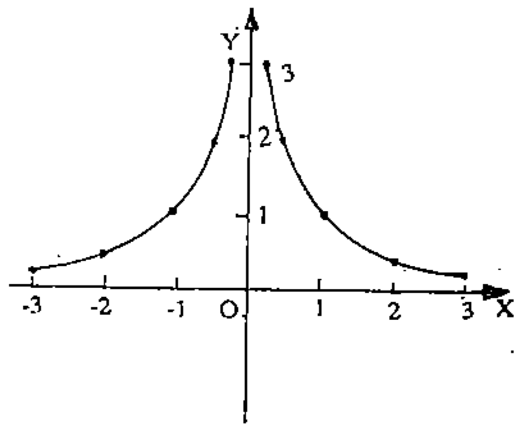


Fig. 4

Sometimes we also need to study the behaviour of a function f(x), as x takes smaller and smaller negative values. This can be examined by the following definition.

Definition 3 A function f is said to tend to a limit L as $x \to -\infty$ if, for each $\varepsilon > 0$, it is possible to choose $K > 0$, such that $|f(x) - L| < \varepsilon$ whenever $x < -K$.

The following example will help you in understanding this idea.

Example 7 Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{1 + x^2}$$

The graph of f is as shown in Fig.5.

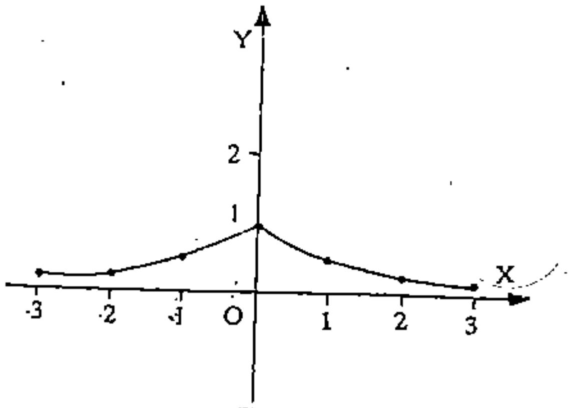


Fig.5

What happens to f(x) as x takes smaller and smaller negative values? Let us make a table (Table 3) to get some idea.

Table 3

x	−10	−100	−1000
$f(x)=\frac{1}{1+x^2}$	1/101	1/10001	1/1000001

We see that as x takes smaller and smaller negative values, f(x) comes closer and closer to zero. In fact $1/(1+x^2) < \varepsilon$ whenever $1 + x^2 > 1/\varepsilon$, that is, whenever $x^2 > (1/\varepsilon) - 1$, that is, whenever either $x < -\left|\frac{1}{\varepsilon}-1\right|^{1/2}$ or $x > \left|\frac{1}{\varepsilon}-1\right|^{1/2}$. Thus, we find that if we take $K=\left|\frac{1}{\varepsilon}-1\right|^{1/2}$, then $x < -K \Rightarrow |f(x)| < \varepsilon$. Consequently, $\lim_{x\to-\infty} f(x) = 0$.

In the above example we also find that $\lim_{x\to\infty} f(x) = 0$.

Let us see how $\lim_{x\to\infty} f(x)$ can be interpreted geometrically.

In the above example we have the function $f(x) = 1/(1 + x^2)$, and as $x \to \infty$, or $x \to -\infty$, $f(x) \to 0$. From Fig. 5 you can see that, as $x \to \infty$ or $x \to -\infty$, the curve $y = f(x)$ comes nearer and nearer the straight line $y = 0$, which is the x-axis.

Similarly, if we say that $\lim_{x\to\infty} g(x) = L$, then it means that, as $x \to \infty$ the curve $y = g(x)$ comes closer and closer to the straight line $y = L$.

Example 8 Let us show that $\lim_{x\to\infty} \frac{x^2}{(1+x^2)} = 1$.

Now, $|x^2/(1 + x^2) - 1| = 1/(1 + x^2)$. In the previous example we have shown that $|1/(1+x^2)| < \varepsilon$ for $x > K$, where $K = |1/\varepsilon - 1|^{1/2}$. Thus, given $\varepsilon > 0$, we choose $K = |1/\varepsilon - 1|^{1/2}$, so that

$$x > K \Rightarrow \left|\frac{x^2}{1+x^2} - 1\right| < \varepsilon.$$ This means that $\lim_{x\to\infty} \frac{x^2}{1+x^2} = 1$.

We show this geometrically in Fig.6.

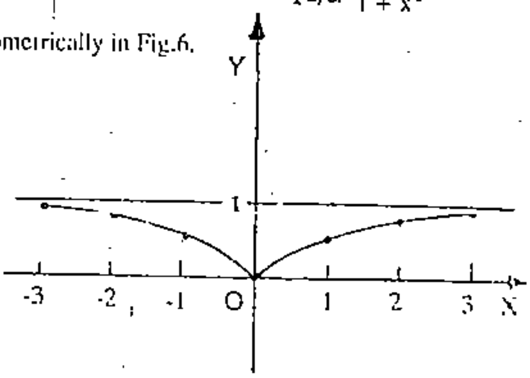


Fig.6

You must be wondering if all the properties given in Theorem 3 also hold when we take limits as $x \to \infty$. Yes, they do.

You can use them to solve this exercise.

E E 4) Show that a) $\lim_{x \to \infty} 1/x = 0$

b) $\lim_{x \to \infty} (1/x + 3/x^2 + 5) = 5$

Sometimes we cannot use Theorem 2 directly, as is clear in the following example. Let us see how to overcome this problem.

Example 9 Suppose, we want to find $\lim_{x \to \infty} \frac{3x + 1}{2x + 5}$.

We cannot apply Theorem 3 directly since the limits of the numerator and the denominator, as $x \to \infty$, cannot be found.

Instead, we rewrite the quotient by multiplying the numerator and denominator by $1/x$, for $x \neq 0$.

Then, $\frac{3x+1}{2x+5} = \frac{3+(1/x)}{2+(5/x)}$, for $x \neq 0$. Now we use Theorem 3 and the fact that $\lim_{x \to \infty} 1/x = 0$ (see E 4 a)), to get

$$\lim_{x \to \infty} \frac{3x+1}{2x+5} = \lim_{x \to \infty} \frac{3+(1/x)}{2+(5/x)}$$

$$= \frac{\lim_{x \to \infty} (3 + 1/x)}{\lim_{x \to \infty} (2 + 5/x)} = \frac{3+0}{2+0} = \frac{3}{2}$$

By now you must be used to the various definitions of limits, so can try this exercise.

E E 5) a) If for some $\varepsilon > 0$, and for every K, $\exists\, x > K$ s.t. $|f(x) - L| > \varepsilon$, what will you infer?

b) If $\lim_{x \to p} f(x) \neq L$, how can you express it in the $\varepsilon - \delta$ form?

We end this section with the following important remark.

Remark 2 In case we have to show that a function f does not tend to a limit L as x approaches p, we shall have to negate the definition of limit (also see E5(b)). Let us see what this means. Suppose we want to prove that $\lim_{x \to p} f(x) \neq L$. Then, we should find some $\varepsilon > 0$ such that for every $\delta > 0$, there is some $x \in]p - \delta, p + \delta[$ for which $| f(x) - L | > \varepsilon$. Through our next example we shall illustrate the negation of the definition of the limit of f(x) as $x \to \infty$.

Example 10 To show that $\lim_{x \to \infty} 1/x \neq 1$, we have to find some $\varepsilon > 0$ such that for any K (howsoever large) we can always find an $x > K$ such that $| 1/x - 1 | > \varepsilon$

Take $\varepsilon = 1/4$. Now, for any $K > 0$, if we take $x = \max \{2, K + 1\}$, we find that $x > K$ and $| 1/x - 1 | > 1/4$. This clearly shows that $\lim_{x \to \infty} 1/x \neq 1$.

2.2.3 One-sided Limits

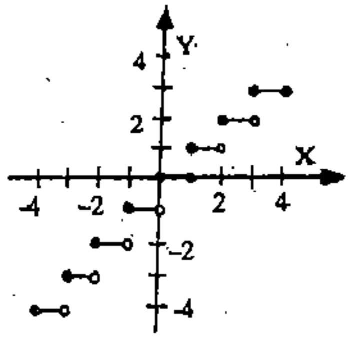


Fig. 7

If we consider the graph of the function $f(x) = [x]$, shown in Fig. 7, we see that f(x) does not seem to approach any fixed value as x approaches 2. But from the graph we can say that if x approaches 2 from the left then f(x) seems to tend to 1. At the same time, if x approaches 2 from the right, then f(x) seems to tend to 2. This means that the limit of f exists if x approaches 2 from only one side (left or right) at a time. This example suggests that we introduce the idea of a one-sided limit.

Definition 4 Let f be a function defined for all x in the interval $]p, q[$. f is said to approach a **limit** L as x **approaches** p **from right** if, given any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $p < x < p + \delta \Rightarrow | f(x) - L | < \varepsilon$.

In symbols we denote this limit by $\lim_{x \to p^+} f(x) = L$

Similarly, the function $f :]a, p[\to \mathbb{R}$ is said to approach a limit L as x **approaches** p **from the left** if, given any $\varepsilon > 0$, $\exists\, \delta > 0$ such that $p - \delta < x < p \Rightarrow | f(x) - L | < \varepsilon$.

This limit is denoted by $\lim_{x \to p^-} f(x)$.

Note that in computing these limits the values of f(x) for x lying on only one side of p are taken into account.

Let us apply this definition to the function $f(x) = [x]$. We know that for $x \in [1, 2[$, $[x] = 1$. That is, [x] is a constant function on $[1, 2[$. Hence $\lim_{x \to 2^-} [x] = 1$. Arguing similarly, we find that since $[x] = 2$ for all $x \in [2, 3[$, [x] is, again, a constant function on $[2, 3[$, and $\lim_{x \to 2^+} [x] = 2$.

Let us improve our understanding of the definition of one-sided limits by looking at some more examples.

Example 11 Let f be defined on $\mathbb{R}$ by setting

$$f(x) = \frac{|x|}{x}, \text{ when } x \neq 0$$
$$f(0) = 0$$

We will show that $\lim_{x \to 0^-} f(x)$ equals -1.

When $x < 0$, $| x | = -x$, and therefore, $f(x) = (-x)/x = -1$. In order to show that $\lim_{x \to 0^-} f(x)$ exists and equals -1, we have to start with any $\varepsilon > 0$ and then find a $\delta > 0$ such that, if $-\delta < x < 0$, then $| f(x) - (-1) | < \varepsilon$.

Since $f(x) = -1$ for all $x < 0$, $| f(x) - (-1) | = 0$ and, hence, any number $\delta > 0$ will work. Therefore, whatever $\delta > 0$ we may choose, if $-\delta < x < 0$, then $| f(x) - (-1) | = 0 < \varepsilon$. Hence $\lim_{x \to 0^-} f(x) = -1$.

Example 12 f is a function defined on $\mathbb{R}$ by setting

$f(x) = x - [x]$, for all $x \in \mathbb{R}$.

Let us examine whether $\lim_{x \to 1^-} f(x)$ exists.

Recall (Unit 1) that this function is given by $f(x) = x$, if $0 \leq x < 1$,

$f(x) = x - 1$ if $1 \leq x < 2$, and, in general

$f(x) = x - n$ if $n \leq x < n + 1$ (see Fig.8)

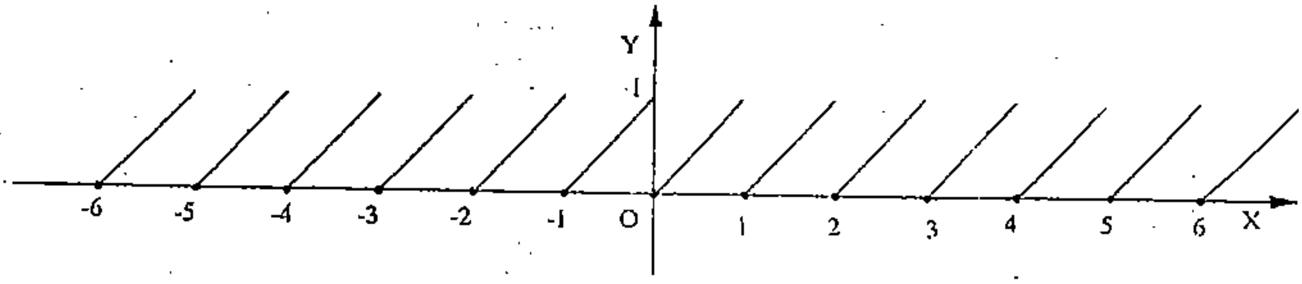


Fig. 8

Since f(x) = x for values of x less than 1 but close to 1, it is reasonable to expect that $\lim_{x \to 1^-} f(x) = 1$. Let us prove this by taking any $\varepsilon > 0$ and choosing $\delta = \min\{1, \varepsilon\}$. We find
$1 - \delta < x < 1 \Rightarrow f(x) = x$ and $|f(x) - 1| = |x - 1| < \delta \le \varepsilon$.

Therefore, $\lim_{x \to 1^-} f(x) = 1$.

Proceeding exactly as above, and noting that f(x) = x − 1 if $1 \le x < 2$, we can similarly prove that $\lim_{x \to 1^+} f(x) = 0$.

E E 6) Prove that a) $\lim_{x \to 3^-} x - [x] = 1$

b) $\lim_{x \to 0^+} \frac{|x|}{x} = 1$

c) $\lim_{x \to 0^-} \frac{(x^2 + 2)|x|}{x} = -2$

Before going further, let us see how the concepts of one-sided limit and limit are connected.

Theorem 4 The following statements are equivalent.

i) $\lim_{x \to p} f(x)$ exists

ii) $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$ exist and are equal.

Proof : To show that i) and ii) are equivalent, we have to show that i) $\Rightarrow$ ii) and ii) $\Rightarrow$ i).

We first prove that i) $\Rightarrow$ ii). For this we assume that $\lim_{x \to p} f(x) = L$. Then given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - L| < \varepsilon$ for $0 < |x - p| < \delta$.

Now, $0 < |x - p| < \delta \Rightarrow p < x < p + \delta$ and $p - \delta < x < p$. Thus, we have $|f(x) - L| < \varepsilon$ for $p < x < p + \delta$ and for $p - \delta < x < p$. This means that $\lim_{x \to p^-} f(x) = L = \lim_{x \to p^+} f(x)$.

We now prove the converse, that is, ii) $\Rightarrow$ i). For this, we assume that

$\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x) = L$. Then, given $\varepsilon > 0$, $\exists\, \delta_1, \delta_2 > 0$

such that

$|f(x) - L| < \varepsilon$ for $p - \delta_1 < x < p$

$|f(x) - L| < \varepsilon$ for $p < x < p + \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for both $p - \delta < x < p$ and $p < x < p + \delta$, we have $|f(x) - L| < \varepsilon$.

This means that $|f(x) - L| < \varepsilon$ whenever $0 < |x - p| < \delta$.

Hence $\lim_{x \to p} f(x) = L$.

Thus, we have shown that i) $\Rightarrow$ ii) and ii) $\Rightarrow$ i), proving that they are equivalent.

From Theorem 4, we can conclude that if $\lim_{x \to p} f(x)$ exists, then $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$ also exist and further

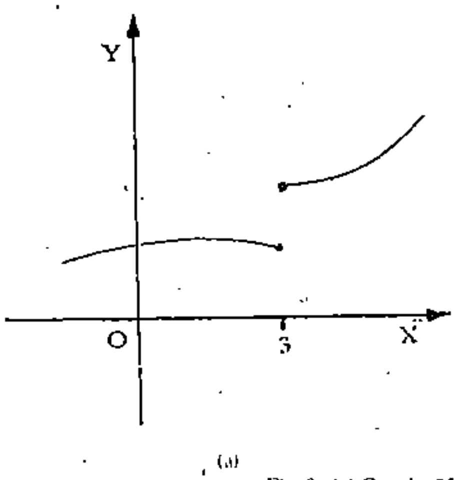
$\lim_{x \to p} f(x) = \lim_{x \to p^+} f(x) = \lim_{x \to p^-} f(x)$

Remark 3 If you apply Theorem 4 to the function $f(x) = x - [x]$ (see Example 12), you will see that $\lim_{x \to 1} \{x - [x]\}$ does not exist as $\lim_{x \to 1^+} \{x - [x]\} \neq \lim_{x \to 1^-} \{x - [x]\}$.

We shall use this concept of one-sided limits to define continuous functions in the next section.

2.3 CONTINUITY

A continuous process is one that goes on smoothly without any abrupt change. Continuity of a function can also be interpreted in a similar way. Look at Fig. 9. The graph of the function f in Fig.9 (a) has an abrupt cut at the point x = 3, whereas the graph of the function g in Fig.9 (b) proceeds smoothly. We say that the function g is continuous, while f is not.



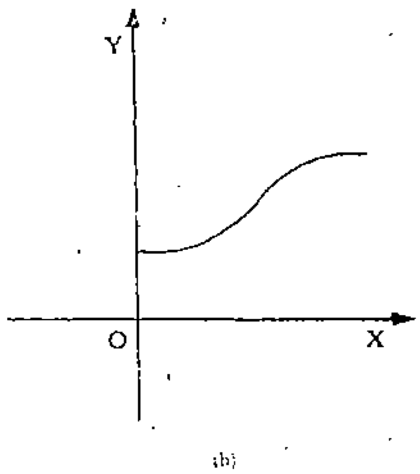


Fig. 9 : (a) Graph of f (b) Graph of g

Continuous functions play a very important role in calculus. As you proceed, you will be able to see that many theorems which we have stated in this course are true only for continuous functions. You will also see that continuity is a necessary condition for the derivability of a function, and that it is a sufficient condition for the integrability of a function. But let us give a precise meaning to "a continuous function" now.

2.3.1 Definitions and Examples

In this section we shall give you the definition and some examples of a continuous function. We shall also give you a short list of conditions which a function must satisfy in order to be continuous at a point.

Definition 5 Let f be a function defined on a domain D, and let r be a positive real number such that the interval $]p-r, p+r[\subset D$. f is said to be continuous at x = p if $\lim_{x\to p} f(x) = f(p)$.

By the definition of limit this means that f is continuous at p if given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $|x - p| < \delta$. To clarify this concept let us look at an example.

Example 13 Let us check the continuity of the function $f : R \to R$ such that f(x) = x at the point x = 0.

Now, f(0) = 0. Thus we want to know if $\lim_{x\to 0} f(x) = 0$.

This is true because given $\varepsilon > 0$, we can choose $\delta = \varepsilon$ and verify that $|x| < \delta \Rightarrow |f(x)| < \varepsilon$. Thus f is continuous at x = 0.

Remark 4 f is continuous at x = p provided the following two criteria are met:

i) $\lim_{x\to p} f(x)$ exists.

ii) $\lim_{x\to p} f(x) = f(p)$.

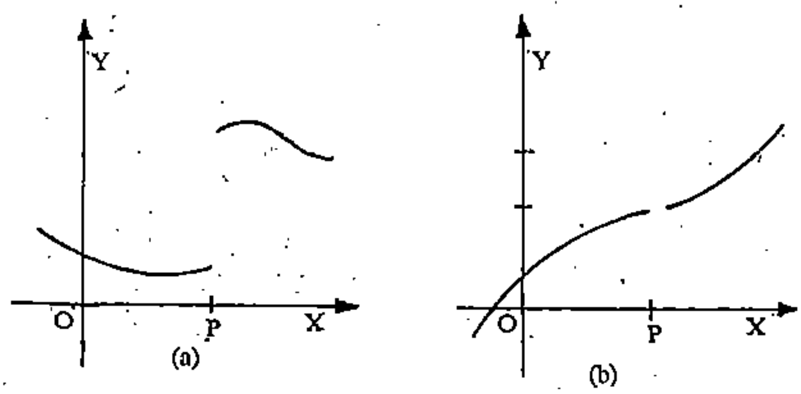


Fig. 10 : (a) Graph of f (b) Graph of g

Fig. 10 shows two discontinuous functions f and g. Criterion i) is not met by f, whereas g fails to meet criterion ii). If you read Remark 3 again, you will find that f(x) = x – [x] is not continuous at x = 1. But we have seen that we can calculate one-sided limits of f(x) = x – [x] at x = 1. This leads us to the following definition.

Definition 6 A function $f :]p, q[\to R$ is said to be continuous from the right at x = p if $\lim_{x\to p^+} f(x) = f(p)$. We say that f is continuous from the left at q if $\lim_{x\to q^-} f(x) = f(q)$.

Thus, f(x) = x – [x] is continuous from the right but not from the left at x = 1 since $\lim_{x\to 1^-} f(x) \neq f(1)$ and $\lim_{x\to 1^+} f(x) = f(1) = 0$.

E E 7) Give $\varepsilon - \delta$ definition of continuity at a point from the right as well as from the left.

E 8) Show that a function $f:]p-r, p+r[\to R$ is continuous at $x = p$ if and only if f is continuous from the right as well as from the left at $x = p$ (Use Theorem 4).

Now that you know how to test the continuity of a function at a point, let us go a step further, and define continuity of a function on a set.

Definition 7 A function f defined on a domain D is said to be **continuous on D**, if it is continuous at every point of D.

Let us see some more examples.

Example 14 Let $f(x) = x^n$ for all $x \in R$ and any $n \in Z^+$. Show that f(x) is continuous at $x = p$ for all $p \in R$.

Z^+ is the set of positive integers.

We know that $\lim_{x \to p} x = p$ for any $p \in R$. Then, by the product rule in Theorem 3, we get

$$\lim_{x \to p} x^n = (\lim_{x \to p} x)(\lim_{x \to p} x) \ldots (\lim_{x \to p} x) \qquad \text{(n times)}$$

$= pp \ldots p$ (n times) $= p^n$. Therefore, $\lim_{x \to p} f(x)$ exists and equals f(p). Hence f is continuous at x = p. Since p was any arbitrary number in R, we can say that f is continuous on R.

Remark 5 Using Example 14 and Theorem 3, we can also prove that polynomial $a_0 + a_1x + \ldots + a_nx^n$, where $a_0, a_1 \ldots a_n \in R$, is continuous on R, that is, $\lim_{x \to p} (a_0 + a_1x + \ldots + a_nx^n) = a_0 + a_1p + \ldots + a_np^n$ for all $p \in R$.

Example 15 The greatest integer function $f: R \to R : f(x) = [x]$ is discontinuous at $x = 2$.

To prove this we recall our discussion in Sec. 2 in which we have proved that $\lim_{x \to 2^-} f(x) = 1$ and $\lim_{x \to 2^+} f(x) = 2$. Thus, since these two limits are not equal, $\lim_{x \to 2} f(x)$ does not exist. Therefore, f is not continuous at x = 2 because the first criterion laid down in Remark 4 is not met.

Example 16 Let $f(x) = |x|$ for all $x \in R$. This f is continuous at x = 0.

Here $f(x) = x$, if $x \geq 0$, and $f(x) = -x$ if $x < 0$. You can show that

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 = f(0)$ and

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0 = f(0)$. Thus, $\lim_{x \to 0} f(x)$ exists and equals f(0). Hence f is continuous at x = 0.

Note : f is also continuous at every other point of R. (Check this statement).

Example 17 Suppose we want to find whether $f(x) = \frac{x^2 - 1}{x - 1}$ is continuous at x = 0.

In Fig.11 you see the graph of f. It is the line $y = x + 1$ except for the point $(1, 2)$.

We can write $f(x) = x + 1$ for all $x \neq 1$.

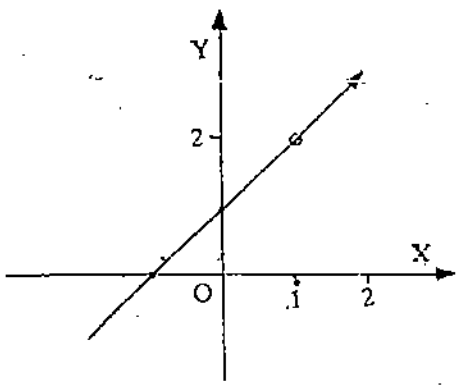


Fig.11

$f(0) = (0^2 - 1)/(0 - 1) = 1$ and

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 1)/(x - 1) = \lim_{x \to 0} (x + 1) = 1$

$\lim_{x \to 0} f(x) = f(0)$, so that f is continuous at $x = 0$.

The exponential function $f(x) = e^x$ and the logarithmic function $f(x) = \ln x$ are continuous functions. You can check this by looking at their graphs in Unit-1. Similarly, $x \to \sin x$ and $x \to \cos x$ are continuous. $x \to \tan x$ is continuous in $]-\pi/2, \pi/2[$. This fact is quite obvious from the graphs of these functions (We have given their graphs in Unit 1). We shall not attempt a rigorous proof of their continuity here.

Caution : Checking the continuity of a function from the smoothness of its graph is not a fool-proof method. If you look at the graph (Fig. 12) of the function $x \to x \sin(1/x)$, you will find that it has no breaks in the neighbourhood of $x = 0$. But this function is not continuous. Observe that the graph oscillates wildly near zero.

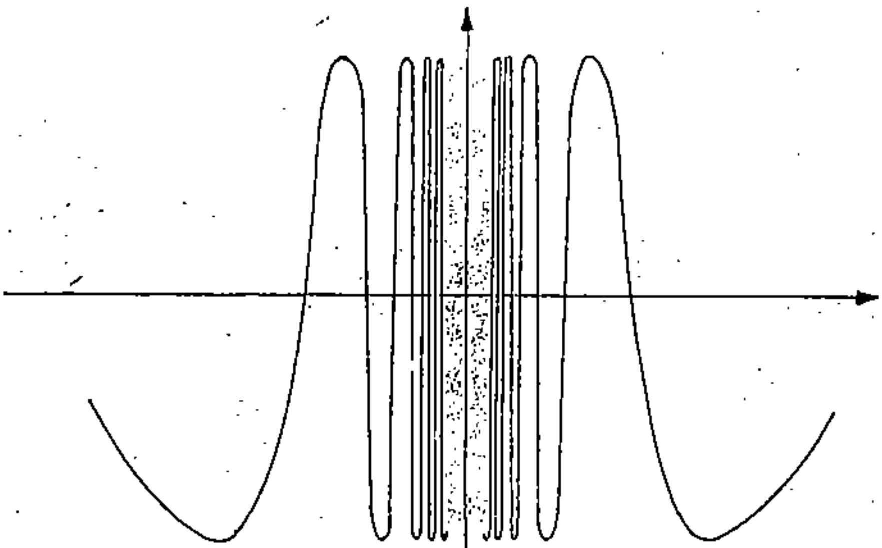


Fig.12

E E 9) Show that the function $f : R \to R$ given by $f(x) = 1/(x^2 - 9)$ is continuous at all points of R except at $x = 3$ and $x = -3$.

Now that we know how to check whether a function is continuous or not, let us go further, and talk about the continuity of some combinations of functions.

2.3.2 Algebra of Continuous Functions

Let f and g be functions defined and continuous on a common domain $D \subseteq R$, and let k be any real number. In Unit 1, we defined the functions f + g, fg, f/g (provided $g(x) \neq 0$ any where in D), kf and | f |. The following theorem tells us about the continuity of these functions.

Theorem 5 Let f and g be functions defined and continuous on a common domain D, and let k be any real number. The functions f + g, kf, | f | and fg are all continuous on D. If $g(x) \neq 0$ anywhere in D, then the function f/g is also continuous on D.

We shall not prove this theorem here.

In Unit 1, you have studied the important concept of composite functions. In Theorem 6, we will talk about the continuity of the composite of two continuous functions. Here again, we shall state the theorem without giving proof as it is beyond the level of this course.

Theorem 6 Let $f : D_1 \to D_2$ and $g : D_2 \to D_3$ be continuous on their domains. Then gof is continuous on D_1. ($D_1, D_2, D_3 \subseteq R$).

Example 18 To prove that $f : R \to R : f(x) = (x^2 + 1)^3$ is continuous at x = 0, we consider the functions $g : R \to R: g(x) = x^3$ and $h : R \to R : h(x) = x^2 + 1$. You can check that f(x) = goh (x). Further, by Remark 5, h is continuous on **R**, and g is also continuous on **R**. Thus, goh = f is continuous on **R**.

Let us see if the converse of the above theorems are true. For example, if f and g are defined on an interval [a, b] and if f + g is continuous on [a, b], does that mean that f and g are continuous on [a, b] ?

No. Consider the functions f and g over the interval [0,1] given by

$$f(x) = \begin{cases} 0, & 0 \le x \le 1/2 \\ 1, & 1/2 < x \le 1 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} 1, & 0 \le x \le 1/2 \\ 0, & 1/2 < x \le 1 \end{cases}$$

Then neither f nor g is continuous at x = 1/2. (Why?) But $(f + g)(x) = 1 \ \forall x \in [0, 1]$. Therefore, f + g is continuous on [0, 1].

Now, if | f | is continuous at a point p, must f also be continuous at p? Again, the answer is No. Take, for example, the function $f : R \to R$ given by

$$f(x) = \begin{cases} -1 & \text{for } x \le 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Then $| f(x) | = 1$ in R and hence | f | is continuous.

But f is not continuous at x = 0 (Why?)

E E 10) If $f : R \to R$ is defined by $f(x) = \begin{cases} 1 & \text{if } x \in Z \\ -1 & \text{if } x \notin Z, \end{cases}$

is f continuous at a) x = 1 b) x = − 3/2?

Before we end this unit, we shall state an important theorem concerning continuous functions. Once again, we won't prove this theorem here. But try to understand its statement because we shall be using it in subsequent units.

Theorem 7 (Intermediate Value Theorem) Let f be continuous on the closed interval [a,b]. Suppose c is a real number lying between f(a) and f(b). (That is, f(a) < c < f(b) or f(a) > c > f(b)). Then there exists some $x_0 \in]a,b[$. such that $f(x_0) = c$.

How can we interpret this geometrically? We have already seen that the graph of a continuous function is smooth. It does not have any breaks or jumps. This theorem says that, if the points (a, f(a)) and (b, f(b)) lie on two opposite sides of a line y = c (see Fig.13), then the graph of f must cross the line y = c.

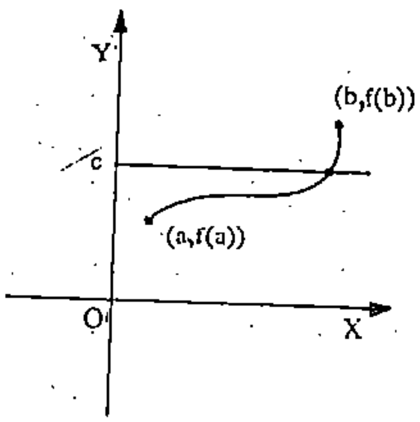


Fig.13

Note that this theorem guarantees only the existence of the number x_0. It does not tell us how to find it. Another thing to note is that this x_0 need not be unique.
That brings us to the end of this unit.

2.4 SUMMARY

We end this unit by summarising what we have covered in it.

1 The limit of a function f at a point p of its domain is L if given $\varepsilon > 0$, $\exists\ \delta > 0$, such that $|f(x) - L| < \varepsilon$ whenever $|x - p| < \delta$.

2 One-sided limits

3 $\lim_{x \to p} f(x)$ exists if and only if
$\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$ both exist and are equal.

4 A function f is continuous at a point x = p if
$\lim_{x \to p} f(x) = f(p)$

5 If the function f and g are continuous on D, then so are the functions f + g, fg, | f |, kf (where $k \in R$) f/g (where $g(x) \neq 0$ in D)

6 The Intermediate Value Theorem : If f is continuous on [a, b] and if f(a) < c < f(b) (or f(a) > c > f(b)), then
$\exists x_0 \in\ ]a, b[$ such that $f(x_0) = c$.

2.5 SOLUTIONS AND ANSWERS

E 1) a) Given any $\varepsilon > 0$, if we choose $\delta = \min\{\varepsilon/2, 1/2\}$, then
$|x - 1| < \delta \leq 1/2 \Rightarrow x > 1/2$,

and $\left|1/x - 1\right| = \left|\frac{x-1}{x}\right| < \left|\frac{x-1}{1/2}\right|$
$= 2|x - 1| < 2\delta < \varepsilon$.

That is, $|x - 1| < \delta \Rightarrow |1/x - 1| < \varepsilon$

Hence, $\lim\limits_{x \to 1} 1/x = 1$

b) $\dfrac{x^3 - 1}{x - 1} - 3 = \dfrac{x^3 - 3x + 2}{x - 1} = (x - 1)(x + 2)$, if $x \neq 1$.

Given $\varepsilon > 0$, if we choose $\delta = \min\{(2/7)\varepsilon, 1/2\}$, then

$|x - 1| < 1/2 \Rightarrow x < 3/2 \Rightarrow x + 2 < 7/2$ and

$$\left|\frac{x^3 - 1}{x - 1} - 3\right| = |(x - 1)(x + 2)| < (7/2)|x - 1| < (7/2) \cdot 2/7 \cdot \varepsilon = \varepsilon.$$

That is, $|x - 1| < \delta \Rightarrow \left|\dfrac{x^3 - 1}{x - 1} - 3\right| < \varepsilon$

Hence, $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1} = 3$.

E 2) $\lim\limits_{x \to 1} 3/x = \dfrac{\lim\limits_{x \to 1} 3}{\lim\limits_{x \to 1} x} = 3/1 = 3$

E 3) $$\lim_{x \to 1} 2x + 5\left(\frac{x^2}{1 + x^2}\right) = 2 \lim_{x \to 1} x + \frac{5 \lim\limits_{x \to 1} x^2}{1 + \lim\limits_{x \to 1} x^2}$$

$$= 2 + \frac{5 \times 1}{1 + 1} = 2 + 5/2 = 9/2$$

E 4) a) Given $\varepsilon > 0$ if we choose $K = 1/\varepsilon$, then

$x > K \Rightarrow |1/x - 0| = |1/x| < 1/K = \varepsilon$.

Thus, $\lim\limits_{x \to \infty} 1/x = 0$

b) Given $\varepsilon > 0$, if we choose $K = 1/\sqrt{\varepsilon}$, then

$x > K \Rightarrow |1/x^2 - 0| = |1/x^2| < 1/K^2 = \varepsilon$

Hence, $\lim\limits_{x \to \infty} 1/x^2 = 0$

Now, $\lim\limits_{x \to \infty} (1/x + 3/x^2 + 5)$

$= \lim\limits_{x \to \infty} 1/x + 3 \lim\limits_{x \to \infty} 1/x^2 + \lim\limits_{x \to \infty} 5 = 0 + 3 \times 0 + 5 = 5$

E 5) a) $\lim\limits_{x \to \infty} f(x) \neq L$

b) $\exists\, \varepsilon > 0$, s.t. $\forall \delta > 0$ $\exists\, x$ s.t. $|x - p| < \delta$ and $|f(x) - L| > \varepsilon$.

E 6) a) Since $x - [x] = x - 2$, $2 \leq x < 3$,

$\lim\limits_{x \to 3^-} x - [x] = \lim\limits_{x \to 3^-} x - 2 = 1$

b) $\lim\limits_{x \to 0^+} |x|/x = \lim\limits_{x \to 0^+} x/x = 1$, $|x| = x$ for $x > 0$.

c) $\lim\limits_{x \to 0^-} \dfrac{(x^2 + 2)|x|}{x} = \lim\limits_{x \to 0^-} \dfrac{(x^2 + 2)(-x)}{x}$ since $|x| = -x$ for $x < 0$

$$= \lim_{x \to 0^-} -(x^2 + 2) = -2$$

E 7) f is continuous from the right at $x = p$ if $\forall \varepsilon > 0$ there exists a $\delta > 0$ s.t.

$p < x < p + \delta \Rightarrow |f(x) - f(p)| < \varepsilon$

f is continuous from the left at $x = p$ if $\forall \varepsilon > 0$ there exists a $\delta > 0$ s.t.

$p - \delta < x < p \Rightarrow |f(x) - f(p)| < \varepsilon$.

E 8) f is continuous at $x = p \Rightarrow \lim\limits_{x \to p} f(x) = f(p)$

$\Rightarrow \lim\limits_{x \to p^+} f(x) = f(p)$ and $\lim\limits_{x \to p^-} f(x) = f(p)$ by Theorem 4

$\Rightarrow$ f is continuous from right and from left at $x = p$.

If f is continuous from right and left,

$\Rightarrow \lim\limits_{x \to p^+} f(x) = f(p) = \lim\limits_{x \to p^-} f(x)$

$\Rightarrow \lim\limits_{x \to p} f(x)$ exists and $= f(p)$

$\Rightarrow$ f is continuous at p.

E 9) $\lim_{x \to a} f(x) = \frac{1}{\lim_{x \to a} x^2 - 9} = \frac{1}{a^2 - 9} = f(a)$ for all a except $a = \pm 3$

Hence f is continuous at all points except at ± 3, f is not defined at ± 3.

E 10) a) f is not continuous at $x = 1$. For $\varepsilon = 1$ and any $\delta > 0$, if x is any non-integer $\in]1-\delta, 1+\delta[$, then

$| f(x) - f(1) | = | -1 - 1 | = 2 > \varepsilon$.

b) f is continuous at $-3/2$. Since given $\varepsilon > 0$, if we choose $\delta < 1/2$, then

$| x - (-3/2) | < 1/2 \Rightarrow -2 < x < -1 \Rightarrow x \notin \mathbf{Z}$ and hence $| f(x) - f(-3/2)| = 0 < \varepsilon$.

UNIT 3 DIFFERENTIATION

Structure

3.1 INTRODUCTION

It was the seventeenth century. Some European mathematicians were working on two basic problems.

i) Is it possible to find the tangent to a given curve at a given point of the curve?

ii) Is it possible to find the area under a given curve?

Two mathematical giants, Newton and Leibniz, independent of each other, solved these problems. The theory that they invented in the process was Calculus.

Newton (1642-1727)

Leibniz (1646-1716)

In this first unit on differentiation, we propose to introduce the concept of a derivative which is a basic tool of calculus. Leibniz was motivated directly by the first problem given above — a problem which was of great significance for scientific applications. He recognised the derivative as the slope of the tangent to the curve at the given point. Newton, on the other hand, arrived at it by considering some physical problems such as determination of the velocity or the acceleration of a particle at a particular instant. He recognised the derivative as a rate of change of physical quantities. We shall now show that both these considerations lead to the concept of derivative as the limit of a ratio. Of course, to understand what a derivative is, you should have gone through Sec. 2 thoroughly.

We shall first differentiate some standard functions using the definition of the derivative. The algebra of derivatives can then be effectively used to write down the derivatives of several functions which are algebraic combinations of these functions. We shall also discuss the chain rule of differentiation which offers an unbelievable simplification in the process of finding derivatives. We shall also establish a relationship between differentiable functions and continuous functions which you have studied in Unit 2.

Objectives

After studying this unit, you should be able to:

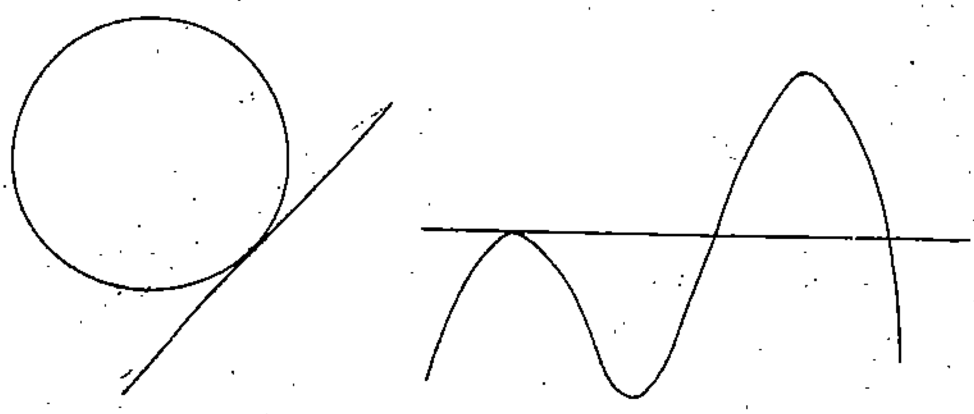
- draw a tangent to a given curve at a given point
- determine the rate of change of a given quantity with respect to another
- obtain the derivatives of some simple functions such as x^n, $|x|$, $\sqrt{x}$, etc. from the first principles
- find the derivatives of functions which can be written as the sum, difference, product, quotient of functions whose derivatives you already know
- derive and use the chain rule of differentiation for writing down the derivatives of a composite of functions
- discuss the relationship between continuity and derivability of a function.

3.2 THE DERIVATIVE OF A FUNCTION

Before defining a derivative, let us consider two problems in the next two subsections. The first is to find the slope of a tangent and the second is to find the rate of change of a given quantity in terms of another.

3.2.1 Slope of a Tangent

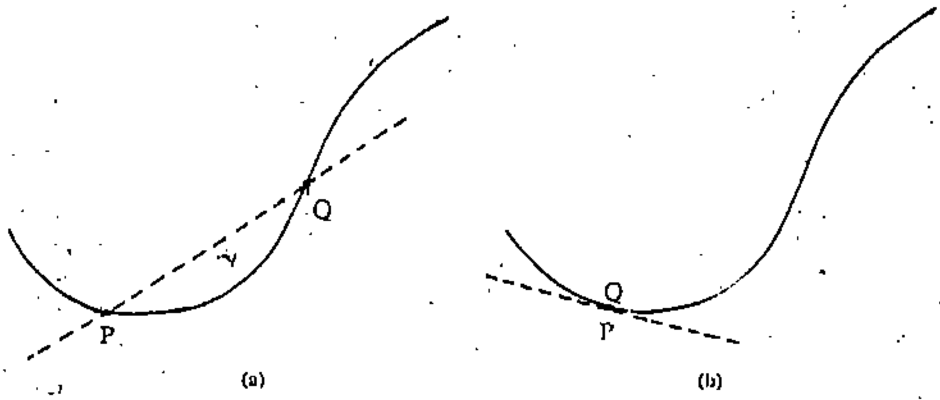
Let us consider the problem of finding a tangent to a given curve at a given point. But, what do we mean by the tangent to a curve? Euclid (300 B.C.) thought of a tangent as a line touching the curve at one point. This definition works fine in the case of a circle Fig.1 (a), but it fails in the case of many other curves (see Fig. 1 (b).



(a) Fig. 1 (b)

We may define a tangent to a curve at P to be a line which best approximates the curve near P. But this definition is still too vague. Then how can we define a tangent precisely? The concept of limit which you have studied in Unit 2 comes to our aid here.

Let P be a fixed point on the curve in Fig. 2 (a), and let Q be a nearby point on that curve. The line through P and Q is called a secant. We define the tangent line at P to be the limiting position (if it exists) of the secant PQ as Q moves towards P along the curve (Fig.2(b)).



(a) (b)

Fig.2

It may not be always possible to find the limiting position of the secant. As we shall see later, there are curves which do not have tangents at some points. In fact, there are curves which do not have a tangent at any point!

There is another question which we can ask here. Suppose we know that a tangent to a curve exists at a point, how do we go about actually drawing the tangent?

The tangent of the angle which a line makes with the positive direction of the x-axis is called the slope of the line.

We have said earlier that the tangent at P is the limiting position of the secant PQ. With reference to a system of coordinate axes OX and OY (Fig.3), we can also say that the tangent at P is a line through P whose slope is the limiting value of the slope of PQ as Q approaches P along the curve. The problem of determining the tangent is, then, the problem of finding the slope of the tangent line.

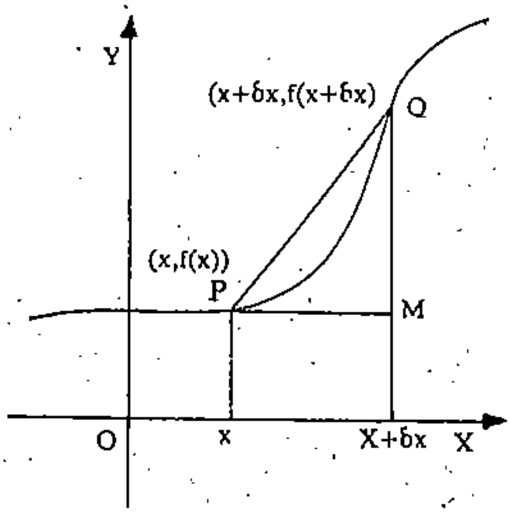


Fig.3

Suppose the curve in Fig.3 is given by $y = f(x)$. Let $(x, f(x))$ be the point P and let Q $(x + \delta x, f(x + \delta x))$ be any other point on the curve. The prefix δ before a variable quantity means a small change in the quantity. Thus, δx means a small change in the variable x. (Caution : δx is one inseparable quantity. It is not $\delta \times x$). The coordinates $(x + \delta x, f(x + \delta x))$ indicate that Q is near P. If θ is the angle which PQ makes with the x-axis, then the slope of PQ = $\tan \theta$ = QM/PM

$$= \frac{f(x + \delta x) - f(x)}{\delta x}$$

The limiting value of $\tan \theta$, as Q tends to P, (and hence $\delta x \to 0$) then gives us the slope of the tangent at P. Thus,

$$\text{the slope of the tangent line} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

This indicates that the tangent line will exist only if the limit of $\frac{f(x + \delta x) - f(x)}{\delta x}$ exists as $\delta x \to 0$.

Remark 1 In Fig.3 we have taken δx to be positive. But our discussion is valid even for negative values of δx.

Let us take an example .

Example 1 Suppose we want to determine the tangent to the parabola $y = x^2$ at the point P(2, 4).

In Fig.4 we give a portion of the parabola in the vicinity of P(2, 4).

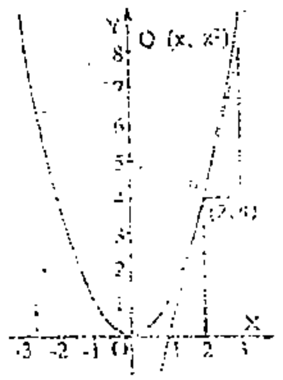


Fig. 4

Equation of a line passing through a point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$

Let Q (x, x^2) be any other point on the parabola. The slope of

$$PQ = \frac{y \text{ coordinate of } Q - y \text{ coordinate of } P}{x \text{ coordinate of } Q - x \text{ coordinate of } P}$$

$$= \frac{x^2 - 4}{x - 2}$$

The tangent at P (2, 4) is the limiting position of PQ as $x \to 2$. Therefore, the slope of the tangent at P is

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

The equation of the tangent line will be $(y - 4) = 4(x - 2)$.

Now, how do we draw this tangent? Just mark the point P′ by moving a distance 1 unit from P, parallel to the x-axis to the right, and then, moving a distance equal to 4 units parallel to the y-axis upward. Join P to P′ as shown in Fig.4. The coordinates of P′ are $(2 + 1, 4 + 4)$. The resulting line will touch the parabola at P, and the slope of the tangent at P = $\tan \theta = 4$.

E E 1) Find the equation of the tangent to the following curves at the given points.

a) $y = 1/x$ at $(2, 1/2)$

b) $y = x^3$ at $(1, 1)$

In this subsection we have given a precise definition of a tangent to a curve. We have also seen how to draw the tangent to a given curve at a given point. Now let us consider the second problem mentioned at the beginning of this section.

3.2.2 Rate of Change

Suppose a particle is moving along a straight line, and covers a distance s in time t. The distance covered depends on the time t. That is $s = f(t)$, a function of time. When the time changes to $t + \delta t$, the distance covered changes from $f(t) = s$ to $f(t + \delta t) = s + \delta s$. Therefore we can say that δs is the distance covered in the time δt. We want to know the average velocity of the particle during the time interval t to $t + \delta t$ (or $t + \delta t$ to t, according as $\delta t > 0$ or $\delta t < 0$).

Now, the average velocity = $\dfrac{\text{Total distance travelled}}{\text{Total time taken}}$

Therefore, the average velocity in the time interval $[t, t + \delta t]$ (or $[t + \delta t, t]$).

$$= \frac{f(t+\delta t) - f(t)}{(t+\delta t) - t} = \frac{(s+\delta s) - s}{(t+\delta t) - t} = \frac{\delta s}{\delta t}, \text{ where}$$

$\delta s = f(t + \delta t) - f(t)$,

But this does not give us the velocity of the particle at a **particular instant** t, which is called **the instantaneous velocity**. How do we calculate this?

To find the velocity at a particular time t, we proceed to find the average velocity in the time interval $[t, t + \delta t]$ (or $[t + \delta t, t]$) for smaller and smaller values of δt.

If δt is very small, then $t + \delta t$ is very near t and so the average velocity during the time interval δt would be very near the velocity at t. It seems reasonable, therefore, to define the instantaneous velocity at time t to be $\lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$.

Thus, we have

$$v = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$$

where $s = f(t)$ is the distance travelled in time t. Comparing this box with the one given at the end of the last subsection, we find that the concepts of the slope of a tangent and the instantaneous velocity are identical. Further, velocity can be considered as the rate of change of distance with respect to time. So, extending our definition of velocity to other rates of change, we can say that if a quantity y depends on x according to the rule $y = f(x)$, then the rate of change of y with respect to x can be defined as

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Example 2 Suppose we want to find the rate of change of the function f defined by $f(x) = x + 5$, $\forall x \in \mathbf{R}$, at $x = 0$.

We shall first calculate the average rate of change of f in an interval $[0, \delta x]$.

This average rate of change of f in $[0, \delta x]$ is

$$\frac{f(0+\delta x) - f(0)}{(0+\delta x) - 0} = \frac{f(\delta x) - f(0)}{\delta x}$$

$$= \frac{\delta x + 5 - 5}{\delta x} = \frac{\delta x}{\delta x} = 1$$

Hence, the rate of change of f at 0, which is the limiting value of this average rate as $\delta x \to 0$,

$$= \lim_{\delta x \to 0} \frac{f(0+\delta x) - f(0)}{\delta x} = \lim_{\delta x \to 0} 1 = 1.$$

Example 3 Suppose a particle is moving along a straight line and the distance s covered in time t is given by the equation $s = (1/2)t^2$. Let us draw the curve represented by the function $s = (1/2)t^2$, measuring time along x-axis and distance along y-axis. Let P and Q be points on the curve which correspond to $t_1 = 2$ and $t_2 = 4$.

We shall show that the average velocity of the particle in the time interval [2,4] is the slope of the line PQ and the velocity at time $t_1 = 2$ is the slope of the tangent to the curve at $t_1 = 2$.

The curve represented by $s = (1/2)t^2$ is a parabola, as shown in Fig. 5. P and Q correspond to the values $t_1 = 2$ and $t_2 = 4$ of t. Now $s_1 = (1/2)\,t_1^2 = 2$ and $S_2 = (1/2)\,t_2^2 = 8$. Therefore, the coordinates of the points P and Q are (2, 2) and (4, 8), respectively.

The slope of $PQ = \frac{8-2}{4-2} = 6/2 = 3$.

Also, the distance travelled by the particle in the time $(t_2 - t_1)$ is $s_2 - s_1 = 8 - 2 = 6$. Therefore, the average velocity of the particle in the time $(t_2 - t_1)$ is

$$\frac{\text{distance travelled}}{\text{time taken}} = 6/2 = 3.$$

δt may be positive or negative

Hence, the slope of PQ is the same as the average velocity of the particle in the time $(t_2 - t_1)$.

Further, to calculate the slope of the tangent at P, we choose a point $R(2 + \delta t, \frac{1}{2}(2 + \delta t)^2)$ on the curve, near P. Then the required slope is $\lim_{\delta t \to 0}$ (slope of PR).

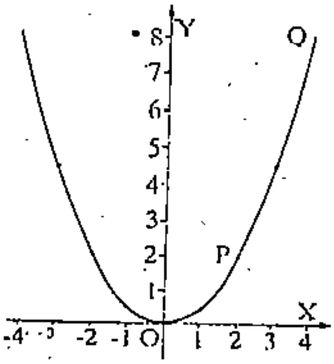


Fig. 5

$$\lim_{\delta t \to 0} \frac{\frac{1}{2}(2+\delta t)^2 - 2}{(2+\delta t) - 2} = \lim_{\delta t \to 0} \frac{\delta t(4+\delta t)}{2\delta t} = 2$$

And, what is the velocity at t_1? It is $\lim_{\delta t \to 0} \delta s/\delta t$, which is again equal to 2. Thus the velocity at t_1 is the same as the slope of the tangent at P.

Remark 2 i) Example 3 is a particular case of the general result: If the path of a particle moving according to $s = f(t)$ is shown in the ts - plane and if P and Q are points on the path which correspond to $t = t_1$ and $t = t_2$, then the average velocity of the particle in time $(t_2 - t_1)$ is given by the slope of PQ and the velocity at time t_1 is given by the slope of the tangent at P.

ii) Distance is always measured in units of length (metres, centimetres, feet) and so velocity v really means v units of distance per unit of time. The slope of the tangent is a dimensionless number, while the velocity has the dimension of length/time.

Now you can try some exercises on your own.

E E 2) A particle is thrown vertically upwards in the air. The distance it covers in time t is given by $s(t) = ut - (1/2)\,gt^2$ where u is the initial velocity and g denotes the acceleration due to gravity. Find the velocity of the particle at any time t.

E E 3) Find the rate of change of the area of a circle with respect to its radius when the radius is 2 cm. (Hint: Express the area of a circle as a function of its radius first).

E 4) Find the average rate of change of the function f defined by $f(x) = 2x^2 + 1, \forall x \in R$ in the interval $[1, 1 + h]$ and hence evaluate the rate of change of f at x = 1.

3.2.3 The Derivative

We have seen that the slope of a tangent and the rate of growth have the same basic concept behind them. Won't it be better, then, to give a separate name to this basic concept, and study it independently of its diverse applications? We give it the name "derivative".

Definition 1 Let y = f(x) be a real-valued function whose domain is a subset D on **R**. Let $x \in D$. If

$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ exists, then it is called the derivative of f at x.

Now, if we write $f(x + \delta x) = y + \delta y$, then derivative of f = $\lim_{\delta x \to 0} \delta y/\delta x$. Here δy denotes the change in y caused by a change δx in x.

The derivative is denoted variously by f'(x), dy/dx or Df. The value of f'(x) at a point x_0 is denoted by $f'(x_0)$. Thus,

The notation dy/dx is due to Leibniz and f'(x) is due to Lagrange (1736 - 1813).

$$f'(x_0) = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$$

If, in the expression $f'(x_0) = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$ we write $x_0 + \delta x = x$, we get $\delta x = x - x_0$, and $\delta x \to 0 \Leftrightarrow x \to x_0$.

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This is an alternative expression for the derivative of f at the point x_0.

Remark 3 In this definition x and y are real numbers and are two dimensionless numbers. If x and y are dimensional quantities (length, time, distance, velocity, area, volume) then the derivative will also have a dimension. For convenience, we shall always treat x and y as dimensionless real numbers. The appropriate dimensions can be added later.

Caution: 'dy' and 'dx' in the expression dy/dx are **not separate entities.** You cannot cancel 'd' from dy/dx to get y/x. The notation only suggests the fact that the derivative is obtained as a ratio.

When f'(x) exists, we say that f is *differentiable* (or derivable) at x. When f is differentiable at each point of its domain D, then f is said to be a **differentiable function.** The process of obtaining the derivative is called **differentiation**. The function f' which associates to each point x of D, the derivative f'(x) at x, is called the **derived function** of f. Thus, the domain of the derived function is $\{x \in D: f'(x) \text{ exists}\}$.

The process of finding the derivative of a function by actually calculating the limit of the ratio $\frac{f(x + \delta x) - f(x)}{\delta x}$ is called differentiating **from first principles.**

As we shall see later, it is not always necessary to find a derivative from the first principles. We shall develop certain rules which can be used to write down the derivatives of some functions without actually finding the limit. Some such rules are contained in the next section.

3.3 DERIVATIVES OF SOME SIMPLE FUNCTIONS

In this section we shall find the derivatives of some simple functions like the constant function, the power function and the absolute value function. We shall illustrate the method of finding the derivative by the first principle method through some examples.

Example 4 Let $f : R \to R$ be a constant function, that is, $f(x) = c$ for all $x \in R$, c being a real number. We shall show that f is differentiable, and its derivative is zero.

Now, $\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{c - c}{\delta x} = \lim_{\delta x \to 0} 0 = 0$

Hence, a constant function is differentiable and its derivative is equal to zero at any point of its domain.

The result of the above example can be seen more easily geometrically. The constant function $f(x) = c \ \forall x \in R$ represents the straight line $y = c$ which is parallel to the x - axis (Fig.6). If we join any two points, P and Q, on it, the line PQ is parallel to the x - axis. Hence, the angle made by PQ with the x - axis is zero. This means the slope of PQ is $\tan 0 = 0$. Since $f'(x)$ is the limit of this slope as $Q \to P$, we get $f'(x) = 0$ for all x in the domain of f.

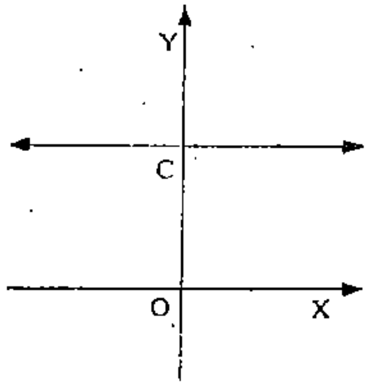


Fig.6.

Example 5 We now show that, if n is a positive integer, then $D(x^n) = nx^{n-1}$.

In order to obtain $D(x^n)$, in case it exists, we have to determine

$$\lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

Notice that we have used the letter h (instead of our usual δx) to denote the small change in the variable x. We are, in fact, free to use any notation; but δx and h are the more commonly used ones.

$$\lim_{h \to 0} \frac{(x + h)^n - x^n}{h},$$

$$= \lim_{h \to 0} \frac{\left(x^n + nhx^{n-1} + \ldots\ldots\ldots + h^n\right) - x^n}{h} \quad \text{(by binomial theorem)}$$

$$= \lim_{h \to 0} \left\{ nx^{n-1} + \frac{n(n-1)}{2} hx^{n-2} + \ldots\ldots\ldots + h^{n-1} \right\}$$

$$= nx^{n-1}$$

The result of the above example is very useful. We shall show later, that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all non-zero $x \in R$ even when n is a negative integer. The result also holds for all $x > 0$ if n is any non-zero real number. Of course, if $n = 0$, then $x^n = 1 \ \forall x$ and hence, $Dx^n = 0$ for all $x \in R$. This means that the result is trivially true for $n = 0$. Nevertheless, right now we are in a position to prove this result for $n = 1/2$. That is,

$\frac{d}{dx}(\sqrt{x}) = \left(\frac{1}{2}\right)x^{-1/2}$, and this we do in the following example.

$\sqrt{x}$ is not defined for $x < 0$.

Example 6 We shall show that the function f defined by $f(x) = \sqrt{x}$, $x > 0$ is differentiable.

We have, $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

The result of our next example is of great significance. Recall that, in Sec. 2 we mentioned that there are functions that have no tangents at some point (or equivalently, have no derivative there). This example will illustrate this fact. Before giving the example we give some definitions.

$Rf'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$, if it exists, is called the **right hand derivative** of f(x) at x = a and is written as $Rf'(a)$. Likewise, $Lf'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$ is called the **left hand derivative** of f(x) at x = a and is written as $Lf'(a)$. If $f'(a)$ exists, we must have $Rf'(a) = Lf'(a) = f'(a)$ (See Unit 2, Theorem 4).

Example 7 The function $f: R \to R$ defined by $f(x) = |x|$ is not derivable at x = 0 but is derivable at every other point of its domain.

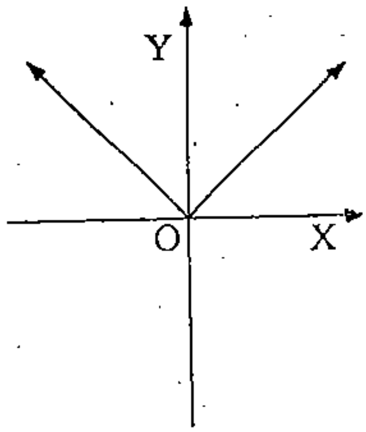


Fig. 7

Fig. 7 shows the graph of this function.

To prove that the given function is not derivable at x = 0, we have to show that

$\lim_{h\to 0} \frac{|0+h|-|0|}{h}$ does not exist. In fact as we shall see, $Rf'(0)$ and $Lf'(0)$ both exist, but they are not equal.

Now $Rf'(0) = \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1$ (since $|h| = h$ for $h > 0$)

And $Lf'(0) = \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1$ (since $|h| = -h$ for $h < 0$)

Therefore, $Rf'(0) = 1 \neq -1 = Lf'(0)$. Hence $f'(0)$ does not exist. We shall now show that the function is derivable at every other point.

First, let $x > 0$. Choose h so that $|h| < x$. This will ensure that $x + h > 0$ whether $h > 0$ or $h < 0$. Now,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|x+h|-|x|}{h}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h}$$

$$= \lim_{h\to 0} h/h = 1$$

Thus f is derivable at x, and $f'(x) = 1$ for all $x > 0$.

You can now complete the solution by solving E 5).

E E 5) Show that $y = |x|$ is derivable and $f'(x) = -1$ at all points $x < 0$.

E E 6) Show that each of the following functions is derivable at $x = 2$. Find $f'(2)$ in each case.

a) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x$

b) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = ax + b$ where a and b are fixed real numbers.

E E 7) Find dy/dx, wherever it exists, for each of the following functions:

a) $y = x^3$ b) $y = |x + 1|$ c) $y = \sqrt{2x + 1}$, $x \geq -\frac{1}{2}$

So far we have obtained derivatives of certain functions by differentiating from the first principles. That is, each time we have calculated $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$

But the process of taking limits is not an easy operation. It can become a very lengthy and complicated affair. In the next section we shall see how to simplify the process of differentiation for some functions.

3.4 ALGEBRA OF DERIVATIVES

Consider the function $f(x) = \frac{2x^3 + 3x^2}{x^4 - 1}$. If we try to find the derivative of this function from the first principles, we will have to do lengthy, complicated calculations. However, a close look at this function reveals that it is composed of several functions: constant functions like 2, 3 and -1, and power functions like x^3, x^2 and x^4. We already know the derivatives of these functions. Can we use this knowledge to find the derivative of f(x)? In this section we shall state and prove some theorems which help us do just that.

3.4.1 Derivative of a Scalar Multiple of a Function

Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function and let $c \in \mathbf{R}$. Then, consider the function $y = cf(x)$. We call this function a scalar multiple of f by c (see Unit 1). The derivative of y with respect to x is

$$\lim_{h \to 0} \frac{(cf)(x+h)-cf(x)}{h} = \lim_{h \to 0} \frac{cf(x+h)-cf(x)}{h}$$

$$= \lim_{h \to 0} c\,\frac{f(x+h)-f(x)}{h}$$

$$= c.\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \quad \text{(by Theorem 3 of Unit 2)}$$

$$= cf'(x)$$

Thus, we have just proved the following theorem.

Theorem 1 If f is a differentiable function and $c \in \mathbf{R}$ then cf is differentiable and $(cf)'(x) = cf'(x)$.

Example 8 To differentiate $y = 7|x|$ we apply the scalar multiple rule obtained in Theorem 1, at all points where the function $|x|$ is differentiable and get

$$\frac{d}{dx}(7|x|) = 7\frac{d}{dx}(|x|)$$

But, in view of Example 7, when $x = 0$, $\frac{d}{dx}(|x|)$ does not exist. When $x > 0$, $\frac{d}{dx}(|x|) = 1$ and when $x < 0$, $\frac{d}{dx}(|x|) = -1$.

Therefore, $\frac{d}{dx}(7|x|) = 7\frac{d}{dx}(|x|)$

$$= \begin{cases} 7 \text{ when } x > 0 \\ -7 \text{ when } x < 0 \end{cases}$$

and $\frac{d}{dx}(7|x|)$ does not exist at $x = 0$.

Note: In example 8 we have used the fact that if $f'(x)$ does not exist at a point then $(cf)'(x)$ also does not exist at that point.

Try the following exercise now.

E 8) Differentiate the following, using Theorem 1.

a) $(5/3)x^3$ b) $8\sqrt{x}$

3.4.2 Derivative of the Sum of Two Functions

$(f+g)(x) = f(x) + g(x)$

Let f and g be differentiable functions from R to R. Let us examine whether f + g, the sum of the functions f and g, is differentiable. Now,

$$\lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$
$$= \lim_{h\to 0} \frac{\{f(x+h) + g(x+h) - f(x) - g(x)\}}{h}$$
$$= \lim_{h\to 0} \left\{ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right\}$$
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$
$$= f'(x) + g'(x)$$

Thus, we have proved the following :

Theorem 2 The sum of two differentiable functions f and g is a differentiable function and $(f+g)'(x) = f'(x) + g'(x)\ \forall\ x \in R$.

The above result can be easily extended to a finite sum, that is,

$$\frac{d}{dx}(f_1 + f_2 + \dots\dots\dots + f_n) = \frac{df_1}{dx} + \frac{df_2}{dx} + \dots\dots\dots + \frac{df_n}{dx},$$

where $f_1, \dots\dots, f_n$ are differentiable functions.

Remark 4 From Theorems 1 and 2 it follows that if f and g are differentiable functions, then f − g is also a differentiable function (since $f - g = f + (-g)$), and $(f-g)'(x) = f'(x) - g'(x)$.

Let us see how Theorem 2 is useful in the following example.

Example 9 To differentiate $3x^2 + 41x - 9$, we apply Theorem 2, and get,

$$\frac{d}{dx}(3x^2 + 41x - 9) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(41x) + \frac{d}{dx}(-9)$$

Now, $\frac{d}{dx}(3x^2) = 3\frac{dx^2}{dx}$ (in view of the theorem)

$= 3 \times 2x = 6x$

$\frac{d}{dx}(41x) = 41\frac{dx}{dx} = 41$

and $\frac{d}{dx}(-9) = 0$ (see Example 4).

Thus, $\frac{d}{dx}(3x^2 + 41x - 9) = 6x + 41$

You are now in a position to solve this exercise.

E 9) Differentiate the following:

a) $5x^3 + 2$

b) $a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$, where $a_i \in R$ for $i = 1, 2, \ldots\ldots, n$.

3.4.3 Derivative of the Product of Two Functions

Let f and g be two differentiable functions on R. We want to find out whether their product fg is also differentiable.

Now, $\lim\limits_{h \to 0} \frac{fg(x+h) - fg(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\{f(x+h) - f(x)\}\,g(x+h) + \{g(x+h) - g(x)\}f(x)}{h}$$

(We have added and subtracted f(x) g(x + h))

$$= \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h}\, g(x+h) \right\} + \lim_{h \to 0} \left\{ \frac{g(x+h) - g(x)}{h}\, f(x) \right\}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x+h) + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \lim_{h \to 0} f(x)$$

(Ref. Unit 2, Theorem 3)

$$= f'(x)\,g(x) + g'(x)\,f(x)$$

Thus, we get the following:

Theorem 3 The product of two differentiable functions is again a differentiable function and its derivative at any point x is given by the formula,

$(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

We can extend this result to the product of three differentiable functions. This gives us

$(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

You see, you have to differentiate only one function at a time. This result can also be extended to the product of any finite number of differentiable functions. Thus, if $f_1, \ldots f_n$ are differentiable functions, then,

$(f_1f_2 \ldots f_n)'(x) = f_1'(x)f_2(x) \ldots f_n(x) + f_1(x)f_2'(x)f_3(x) \ldots f_n(x) + \ldots + f_1(x)f_2(x) \ldots f_n'(x).$

Theorem 3 is very useful in simplifying calculations, as you can see in the following example.

Example 10 To differentiate $f(x) = x^2(x + 4)$, we take $g(x) = x^2$, $h(x) = x + 4$. We have,

$f(x) = x^2(x + 4) = g(x)h(x)$

Now, $g'(x) = \frac{d}{dx}(x^2) = 2x$ and $h'(x) = \frac{d}{dx}(x + 4) = 1$.

Thus, $f'(x) = g'(x)h(x) + h'(x)g(x)$

$$= 2x(x+4) + 1 \times x^2$$

$$= 2x^2 + 8x + x^2 = 3x^2 + 8x$$

Remark 5 You could also have differentiated $x^2(x+4)$ without using Theorem 3, as follows:

$x^2(x+4) = x^3 + 4x^2$

Therefore, $\frac{d}{dx}(x^2(x+4)) = \frac{d}{dx}(x^3 + 4x^2)$

$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(4x^2) \text{ (By Theorem 2)}$$

$$= 3x^2 + 4(2x) = 3x^2 + 8x$$

This shows that the same function can be differentiated by using different methods. You may use any method that you find convenient. This observation should also help you to check the correctness of your result. (We assume that you would not make the same mistake while using two different methods!)

E 10) Using Theorem 3, differentiate the following functions. Also, differentiate these functions without using Theorem 3, and compare the results.

a) $x\sqrt{x}$ b) $(x^5 + 2x^3 + 5)^2$ c) $(x+1)(x+2)(x+3)$

3.4.4 Derivative of the Quotient of Two Functions

Let $\phi = f/g$, where f and g are differentiable functions on R, and $g(x) \neq 0$ for any x. Then,

$$\lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = \lim_{h \to 0} \frac{(f/g)(x+h) - (f/g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \lim_{h \to 0} \frac{g(x)\, f(x+h) - f(x)\, g(x+h)}{hg(x)\, g(x+h)}$$

$$= \lim_{h \to 0} \frac{g(x)\,\frac{f(x+h) - f(x)}{h} - f(x)\,\frac{g(x+h) - g(x)}{h}}{g(x+h)\, g(x)}$$

(by adding and subtracting $f(x)g(x)$ from the numerator)

$$= \frac{\lim_{h\to 0}\left\{ g(x)\cdot\frac{f(x+h)-f(x)}{h} - f(x)\,\frac{g(x+h)-g(x)}{h}\right\}}{\lim_{h\to 0} g(x+h) - g(x)}$$

$$= \frac{\lim_{h\to 0}\left[g(x)\left\{\frac{f(x+h)-f(x)}{h}\right\}\right] - \lim_{h\to 0}\left[f(x)\left\{\frac{g(x+h)-g(x)}{h}\right\}\right]}{\lim_{h\to 0} g(x+h)\,\lim_{h\to 0} g(x)}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Thus, we get the following.

Theorem 4 The quotient f/g of two differentiable functions f and g such that $g(x) \neq 0$, for any x in its domain, is again a differentiable function and its derivative at any point x is given by the following formula :

$$(f/g)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

This can also be written as

$$\frac{d}{dx}\left(\frac{\text{numerator}}{\text{denominator}}\right) = \frac{(\text{denominator})\ (\text{derivative of numerator}) - (\text{numerator})\ (\text{derivative of denominator})}{(\text{denominator})^2}$$

We will obtain an important corollary to Theorem 4 now.

Corollary 1 If g is a function such that $g(x) \neq 0$ for any x in its domain, then

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right)' = \frac{-g'(x)}{(g(x))^2}$$

Proof : In the result of Theorem 4, take f to be the constant function 1. Then $f'(x) = 0$ for all x.

Therefore,

$$\left(\frac{1}{g(x)}\right) = \left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}, \text{ where } f(x) = 1.$$

$$= \frac{g(x)\times 0 - 1\times g'(x)}{(g(x))^2} = \frac{-g'(x)}{(g(x))^2}$$

Example 11 We shall now show that $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a negative integer and $x \neq 0$. We have already proved this result for a positive integer n in Example 5.

Consider the function $f : \mathbf{R}\setminus\{0\} \to \mathbf{R}$ given by $f(x) = x^{-m}$, where $m \in \mathbf{N}$. Then $f(x) = 1/x^m\ \forall x \in \mathbf{R}$. Thus, $f = 1/g$, where $g(x) = x^m$ for all $x \in \mathbf{R}$, $x \neq 0$. g is a differentiable function and $g(x) \neq 0$ if $x \neq 0$. So, except at $x = 0$, we find that

$$f'(x) = \frac{-g'(x)}{(g(x))^2} \text{ (from Corollary 1)}$$

$$= \frac{-mx^{m-1}}{(x^m)^2} \quad (g'(x) = mx^{m-1} \text{ by using Example 5})$$

$$= \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1}$$

Denoting $-m$ by n, we get $f(x) = x^n$, and $f'(x) = nx^{n-1}$

Example 12 Let us differentiate the function f given by $f(x) = (x^{-2} + 2) / (x^2 + 2x)$

We can write f as the quotient g/h where $g(x) = (x^{-2} + 2)$ and $h(x) = x^2 + 2x$.

Now, $g'(x) = \frac{d}{dx}(x^{-2}) + \frac{d}{dx}(2) = -2x^{-3} + 0 = \frac{-2}{x^3}$

Also $h'(x) = 2x + 2$.

$$\text{Therefore, } f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

$$= \frac{(x^2 + 2x)(-2/x^3) - (x^{-2} + 2)(2x + 2)}{(x^2 + 2x)^2}$$

$$= \frac{-4x^{-1} - 6x^{-2} - 4x - 4}{(x^2 + 2x)^2}$$

$$= \frac{-(4x + 4 + 4x^{-1} + 6x^{-2})}{(x^2 + 2x)^2}$$

E E 11) Differentiate

a) $\frac{2x+1}{x+5}$ b) $\frac{1}{a + bx + cx^2 + dx^3}$ where a, b, c, d are fixed real numbers

c) $\frac{2x^3 + 3x^2}{x^4 - 1}$

E E 12) Obtain the derivative of $1/f(x)$ by differentiating from first principles, assuming that $f(x) \neq 0$ for any x.

E 13) Differentiate $f(x) = \frac{2 + 5x + 7x^{-1}}{x^5}$ by three different methods.

3.4.5 The Chain Rule of Differentiation

The chain rule of differentiation is a rule for differentiating a composite of functions (Ref. Unit 1). It is a remarkable rule which helps us to differentiate complicated functions in an easy and elegant way.

We establish the rule in the following theorem.

Theorem 5 Let y = g(u) and u = f(x). If both dy/du and du/dx exist, then dy/dx exists and is given by $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Proof: We first note that y = g(u) = gf(x) = (gof) (x), so that y is the composite function gof. We are given that y, regarded as a function of u, is differentiable. We want to prove that y, regarded as a function of x, is also differentiable. To do this we must show that $\lim_{\delta x \to 0} \delta y/\delta x$ exists, where δy is the change in the variable y corresponding to a change δx in the variable x. Now, δu, the change in the value of u corresponding to a change δx in the value of x, is given by δu = f(x + δx) – f(x).

We have $\lim_{\delta x \to 0} \delta u = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} \delta x\right)$

$$= \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \times \lim_{\delta x \to 0} \delta x$$

$$= \frac{du}{dx} \times 0 = 0$$

This means that δu → 0 as δx → 0

We assume however that δu ≠ 0.

This implies that $\lim_{\delta x \to 0} \frac{\delta y}{\delta u} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u}$ exists and is equal to $\frac{dy}{du}$

Now, $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$, and we know that

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta u} = \frac{dy}{du} \text{ and } \lim_{\delta x \to 0} \frac{\delta u}{\delta x} = \frac{du}{dx}$$

Hence, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

Hence dy/dx exists and is equal to dy/du × du/dx.

You may find it more convenient to remember and use the rule in the following form:

If $h(x) = g(f(x))$ is the composite of two differentiable functions g and f, then h is differentiable and $h'(x) = g'(f(x))f'(x)$.

To clarify this rule let us look at the following example

Example 13 Here we shall differentiate $y = (2x + 1)^3$ with respect to x.

Let $u = 2x + 1$. Then $y = (2x + 1)^3 = u^3$.

Now y is a differentiable function of u and u is a differentiable function of x. $dy/du = 3u^2$ and $du/dx = 2$. Hence we can use the chain rule to get $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= 3u^2 \cdot 2 = 6u^2 = 6(2x + 1)^2$$

You might be thinking that there was really no necessity of using the chain rule here. We could simply expand $(2x + 1)^3$ and then write the derivative. But the situation is not always as simple as in this example. You would appreciate the power of the chain rule after using it in the next example.

Example 14 To differentiate $(x^3 + 2x^2 - 1)^{100}$,

let $y = (x^3 + 2x^2 - 1)^{100}$ and let $u = (x^3 + 2x^2 - 1)$. Then $y = u^{100}$

Since dy/du and du/dx both exist, and $dy/du = 100u^{99}$ and $du/dx = 3x^2 + 4x$, therefore, by chain rule, dy/dx = dy/du. du/dx.

$$= 100\,u^{99} \cdot (3x^2 + 4x)$$

$$= 100\,(x^3 + 2x^2 - 1)^{99}\,(3x^2 + 4x)$$

Can you really attempt to solve the above example without using the chain rule? Don't you think the rule has simplified matters a lot for you?

Instead of introducing u explicitly each time while applying the chain rule, after a little practice you would find it more convenient to do away with u and arrange the working in the above example as follows:

$$\frac{dy}{dx} = \frac{d(x^3 + 2x^2 - 1)^{100}}{d(x^3 + 2x^2 - 1)} \cdot \frac{d}{dx}(x^3 + 2x^2 - 1)$$

$$= 100\,(x^3 + 2x^2 - 1)^{100-1} \cdot (3x^2 + 4x)$$

$$= 100\,(x^3 + 2x^2 - 1)^{99}\,(3x^2 + 4x)$$

Our next example illustrates that this rule can be extended to three functions.

Example 15 To differentiate $\{(5x + 2)^2 + 3\}^4$, we write $y = \{(5x + 2)^2 + 3\}^4$, $u = (5x + 2)^2 + 3$ and $v = 5x + 2$

Then $y = u^4$ and $u = v^2 + 3$. That is, y is a function of u, u is a function of v, and v is function of x. By extending the chain rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

This gives,

$$dy/dx = 4u^3 \cdot 2v \cdot 5 = 40u^3 v$$

$$= 40\{(5x + 2)^2 + 3\}^3 (5x + 2)$$

This example illustrates that there may be situations in which we may go on using chain rule for a function of a function of a function ..., and so on. This perhaps justifies the name 'chain' rule. Thus, if $g_1 \ldots g_n$ and h are functions such that $h = (g_1 o g_2 o \ldots o g_n)(x)$, then

$$h'(x) = g_1'(g_2 \circ \ldots \circ g_n(x))\, g_2'(g_3 \circ \ldots \circ g_n)(x) \ldots g_{n-1}'(g_n)(x) \cdot g_n'(x)$$

E 14) Find dy/dx for each of the following using the chain rule:

a) $\dfrac{5}{1 + 5x + 7x^2}$ b) $\dfrac{(2x+3)^2}{1 + (2x+3)^3}$ c) $\left\{(9x+5)^3 + (9x+5)^{-3}\right\}^7$

3.5 CONTINUITY VERSUS DERIVABILITY

We end this unit with the relationship of differentiability with continuity, which we have studied in Unit 2. In Sec. 3 of Unit 2 we proved that the function $y = |x|$ is continuous $\forall x \in \mathbf{R}$. We have also proved that this function is derivable at every point except at $x = 0$ in Example 7. This means that the function $y = |x|$ is continuous at $x = 0$, but is not derivable at that point. Thus, this shows that a function can be continuous at a point without being derivable at that point. However, we will now prove that if a function is derivable at a point, then it must be continuous at that point; or derivability $\Rightarrow$ continuity.

Recall that a function f is said to be continuous at a point x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

Theorem 6 Let f be a function defined on an interval I. If f is derivable at a point $x_0 \in I$, then it is continuous at x_0.

Proof:
If $x \neq x_0$ then we may write $f(x) - f(x_0) = \dfrac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$

Since f is derivable at x_0, $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists and equals $f'(x_0)$.

Thus, taking limits as $x \to x_0$, we have,

$$\lim_{x \to x_0} [f(x) - f(x_0)]$$

$$= \lim_{x \to x_0} \left\{ \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right\}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0)$$

$$= f'(x_0) \times 0 = 0$$

Therefore, $\lim_{x \to x_0} f(x) - \lim_{x \to x_0} f(x_0) = 0$

That is, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} f(x_0) = f(x_0)$

Consequently, f is continuous at x_0.

As we have seen, the function $y = |x|$ is continuous but not derivable at only one point, $x = 0$. But there are some continuous functions which are not derivable at infinitely many points. For instance, look at Fig. 8.

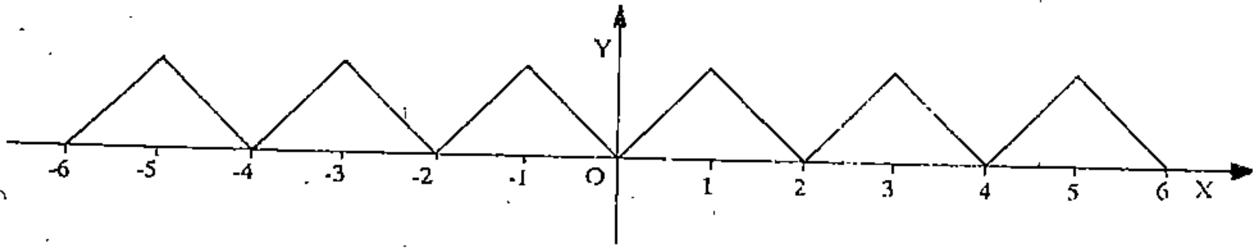


Fig. 8

It shows the graph of a continuous function which is not derivable at infinitely many points. Can you mark those infinitely many points at which this function is not derivable? You can take your hint from the graph of the function $y = |x|$.

The situation is, in fact, much worse. There are functions which are **continuous everywhere but differentiable nowhere.** The discovery came as a surprise to the nineteenth century mathematicians who believed, till then, that if a function is so bad that it is not derivable at any point, then it can't be so good that it is continuous at every point. The first such function was put forth by Weierstrass (although he is said to have attributed the discovery to Riemann) in 1812. He showed that the function f given by $f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$, where a is an odd integer and b is a positive constant between 0 and 1 such that $ab > 1 + 3\pi/2$, is a function which is continuous everywhere, but derivable nowhere. It will not be possible for us to prove this assertion at this stage.

Sometimes we use Theorem 6 to prove that a given function is continuous at a given point. We prove that its derivative exists at that point. By Theorem 6 then, the continuity automatically follows.

E E 15) Is the function $f : [0,1] \to R$; $f(x) = \frac{(2x+3)^{50}}{9x+2}$ continuous at $x = 0.1$?

3.6 SUMMARY

We conclude this unit by summarising what we have covered in it.

1 For any function $y = f(x)$

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$ (if it exists) is called the derivative of f at x, denoted by $f'(x)$.

The function f' is the derived function. The derivative $f'(x)$ is the slope of the tangent to the curve $y = f(x)$ at the point (x, y). The derivative also gives the rate of change of the function with respect to the independent variable.

2 The derivative of a constant function is 0.

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

where n is any integer (and $x \neq 0$ if $n < 0$).

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

3 The function $y = |x|$ is derivable at every point except at $x = 0$.

4 $(cf)' = cf'$, c a constant.

$(f + g)' = f' + g'$

$(f - g)' = f' - g'$

$(fg)' = fg' + gf'$

$(f/g)' = \dfrac{g'f - f'g}{g^2}$

$(gof)' = g'(f).f'$

5 Every derivable function is continuous. The converse is not true, that is, there exist functions which are continuous but not differentiable.

3.7 SOLUTIONS AND ANSWERS

E 1) a) $\left.\frac{dy}{dx}\right|_{x=2} = -1/4$. Equation : $(y - 1/2) = (-1/4)(x - 2)$ or $x + 4y = 4$

b) $\left.\frac{dy}{dx}\right|_{x=1} = 3$. Equation : $(y - 1) = 3(x - 1)$

E 2) $v = ds/dt = u - gt$

E 3) area $= A = \pi r^2$. $\left.\frac{dA}{dr}\right|_{r=2} = 4\pi$

E 4) average rate of change of f in $[1, 1 + h]$ =

$$\frac{f(1 + h) - f(1)}{h} = \frac{2(1 + h)^2 + 1 - (2 \times 1^2 + 1)}{h} = 4 + 2h$$

rate of change of f at $x = 1 = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}$, where h may be positive or negative.

$$= \lim_{h \to 0}(4 + 2h) = 4$$

E 5) If $x < 0$, choose $0 < h < |x|$ then $x + h < 0$ and

$$\lim_{h \to 0} \frac{|x + h| - |x|}{h} = \frac{-(x + h) - (-x)}{h} = \frac{-h}{h} = -1$$

Thus $f'(x) = -1$ if $x < 0$. f is derivable for $x < 0$.

E 6) a) $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{2 + h - 2}{h} = 1$

b) $f'(2) = \lim_{h \to 0} \frac{a(2+h) + b - (a \times 2 + b)}{h}$

$$= \lim_{h \to 0} \frac{ah}{h} = a.$$

E 7) a) $\frac{dy}{dx} = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$

b) If $x > -1$, $x + 1 > 0$, choose $h > 0$ s.t. $h < |x + 1|$.

then $x + h + 1 > 0$ and $\lim_{h \to 0} \frac{|x + h + 1| - |x + 1|}{h}$

$$= \lim_{h \to 0} \frac{(x + h + 1) - (x + 1)}{h} = 1.$$

If $x < -1$, $\lim_{h \to 0} \frac{|x + h + 1| - |x + 1|}{h} = -1$.

Thus dy/dx exists when $x > -1$ or when $x < -1$. It does not exist when $x = -1$ since $Rf'(-1) = 1$ and $Lf'(-1) = -1$.

c) $\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{2(x + h) + 1} - \sqrt{2x + 1}}{h} \times \frac{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}}{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}}$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}} = \frac{1}{\sqrt{2x + 1}}$$

E 8) a) $\frac{d}{dx}\left(\frac{5}{3} x^3\right) = \frac{5}{3} \frac{d}{dx}(x^3) = \frac{5}{3} \times 3x^2 = 5x^2$

b) $\frac{d}{dx}\left(8\sqrt{x}\right) = 8\frac{d}{dx}\left(\sqrt{x}\right) = \frac{4}{\sqrt{x}}$

E 9) a) $15x^2$ b) $a_1 + 2a_2x + \ldots + na_nx^{n-1}$

E 10) a) $\frac{d}{dx}\left(x\sqrt{x}\right) = (3/2)\sqrt{x} = x\left(\frac{1}{2}x^{-1/2}\right) + \sqrt{x}$

b) $\frac{d}{dx}[(x^5 + 2x^3 + 5)(x^5 + 2x^3 + 5)]$

$= (x^5 + 2x^3 + 5)(5x^4 + 6x^2) + (5x^4 + 6x^2)(x^5 + 2x^3 + 5)$

$= 2(x^5 + 2x^3 + 5)(5x^4 + 6x^2)$

c) $dy/dx = (x + 2)(x + 3) + (x + 1)(x + 3) + (x + 1)(x + 2)$

E 11) a) $\frac{2(x + 5) - (2x + 1)}{(x + 5)^2} = \frac{9}{(x + 5)^2}$

b) $\frac{-(b + 2cx + 3dx^2)}{(a + bx + cx^2 + dx^3)^2}$

c) $\frac{(x^4 - 1)(6x^2 + 6x) - (2x^3 + 3x^2)(4x^3)}{(x^4 - 1)^2}$

$$= \frac{6x(x + 1)(x^4 - 1) - 4x^5(2x + 3)}{(x^4 - 1)^2}$$

E 12) $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \lim_{h \to 0} \frac{\frac{1}{f(x + h)} - \frac{1}{f(x)}}{h}$

$$= \lim_{h \to 0} \frac{1}{h} \frac{f(x) - f(x+h)}{f(x) f(x+h)}$$

$$= \frac{\lim_{h \to 0} \frac{f(x) - f(x+h)}{h}}{f(x) \lim_{h \to 0} f(x+h)}$$

$$= \frac{-f'(x)}{f(x)^2}$$

E 13) a) $f(x) = 2x^{-5} + 5x^{-4} + 7x^{-6}$

$f'(x) = -10x^{-6} - 20x^{-5} - 42x^{-7}$

b) $f'(x) = \frac{x^5(5 - 7x^{-2}) - 5x^4(2 + 5x + 7x^{-1})}{x^{10}}$

$= x^{-10}(-20x^5 - 42x^3 - 10x^4)$

$= -20x^{-5} - 42x^{-7} - 10x^{-6}$

c) $f(x) = x^{-5}(2 + 5x + 7x^{-1})$

$f'(x) = x^{-5}(5 - 7x^{-2}) - 5x^{-6}(2 + 5x + 7x^{-1})$

$= -20x^{-5} - 42x^{-7} - 10x^{-6}$

E 14) a) $u = 1 + 5x + 7x^2$, $y = u^{-5}$

$dy/dx = (-5/u^6)(5 + 14x) = \frac{-25 - 70x}{(1 + 5x + 7x^2)^6}$

b) $u = 2x + 3$, $y = u^2/(1 + u^3)$

$$\frac{dy}{dx} = \frac{2u(1 + u^3) - 3u^4}{(1 + u^3)^2} \times 2$$

$$= \frac{4(2x+3)\{1 + (2x+3)^3\} - 6(2x+3)^4}{[1 + (2x+3)^3]^2}$$

c) $u = 9x + 5$, $v = u^3 + u^{-3}$, $y = v^7$

$dy/dx = 7v^6(3u^2 - 3u^{-4}) \times 9$

$= 63\,[(9x+5)^3 + (9x+5)^{-3}]^6\,[3(9x+5)^2 - 3(9x+5)^{-4}]$

E 15) $f'(x) = \frac{50(2x+3)^{49}\, 2(9x+2) - 9(2x+3)^{50}}{(9x+2)^2}$

exists at $x = 0.1$. Hence the function is continuous at $x = 0.1$.

UNIT 4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Structure

4.1 INTRODUCTION

In Unit 3 we have introduced the concept of derivatives. We have also talked about the algebra of derivatives and the chain rule which help us in calculating the derivatives of some complex functions. This unit will take you a step further in your study of differential calculus.

In this unit we shall first find the derivatives of standard trigonometric functions. We shall then go on to study the inverse function theorem and its applications in finding the derivatives of inverses of some standard functions. Finally, we shall see how the use of transformations can simplify the problem of differentiating some functions.

Objectives

After reading this unit you should be able to:

- find the derivatives of trigonometric functions
- state and prove the inverse function theorem
- use the inverse function theorem to find the derivatives of inverse trigonometric functions
- use suitable transformations to differentiate given functions.

4.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

In this section we shall calculate the derivatives of the six trigonometric functions: sinx; cos x, tan x, cot x, sec x and cosec x. You already know that these six functions are related to each other. For example, we have:

i) $\sin^2 x + \cos^2 x = 1$ ii) tan x = sin x/cos x, and many more identities which express the relationships between these functions. As you will soon see, our job of finding the derivatives of all trigonometric functions becomes a lot easier because of these identities. But let us first evaluate some important limits which will prove to be very useful later.

4.2.1 Some Useful Limits

In the next subsection we shall come across $\lim_{t \to 0} \frac{\sin t}{t}$ and $\lim_{t \to 0} \sin t$. So let's try to calculate these. For this, we first assume that $0 < t < \pi/2$ and consider a circle with radius 1 unit, given by $x^2 + y^2 = 1$ as shown in Fig. 1.

The line OT passes through the origin and has slope = tan t. Therefore, we can write its equation as y = x tan t. This means that the y-coordinate of the point T is tan t, since its x-coordinate is 1.

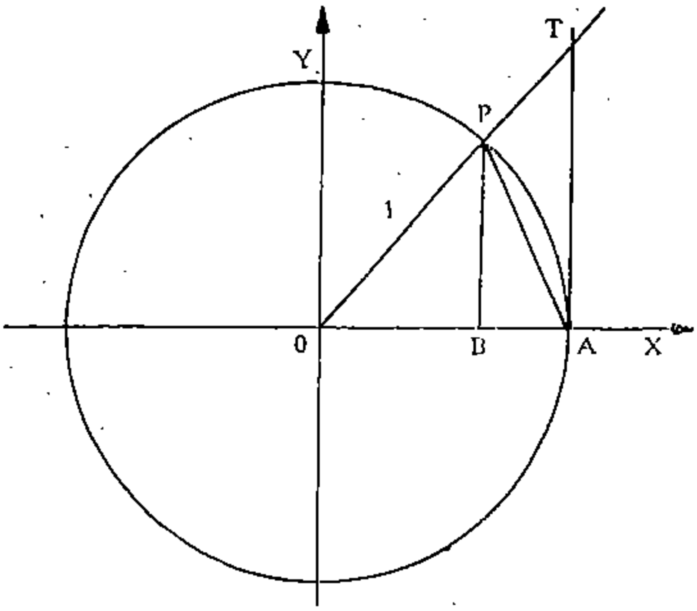


Fig. 1

From the figure we can see that

area of ΔOPA < area of sector OPA < area of ΔOTA . . . . . (1)

Now, the area of $\Delta OPA = \frac{1}{2} \times 1 \times PB = \frac{1}{2} \sin t$,

The area of sector OPA $= \frac{1}{2} \times 1 \times t = \frac{1}{2} t$,

The area of OTA $= \frac{1}{2} \times 1 \times \tan t$

If the sectorial angle is θ, the area of a sector of a circle of radius r is $(1/2)r^2\theta$

Thus, inequality (1) can be written as:

$\sin t < t < \tan t$ . . . . . (2)

Since $0 < t < \pi/2$, $\sin t > 0$, therefore, from the left-hand inequality in (2) we get

$0 < \sin t < t$ . . . . . (3)

Now, if $-\pi/2 < t < 0$, then $0 < -t < \pi/2$, and applying (3) to $-t$, $0 < \sin(-t) < -t$ or $0 < -\sin t < -t$ since $\sin(-t) = -\sin t$. This means that if $-\pi/2 < t < 0$, then $t < \sin t < 0$. . .(4)

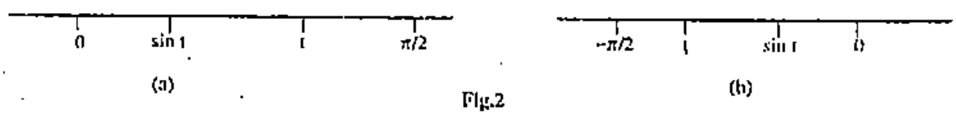


Fig.2

In Fig. 2(a) and (b) you can see the representation of (3) and (4), respectively.

We can combine (3) and (4) and write

$-|t| < \sin t < |t|$ for $-\pi/2 < t < \pi/2$, $t \neq 0$.

Here we are using various results about the order relation from Unit 1.

You have seen in Unit 2 that $\lim_{t \to 0} |t| = 0$. From this we can also say that $\lim_{t \to 0} -|t| = 0$.

Now applying the sandwich theorem (Theorem 2 of Unit 2) to the functions $-|t|$, $\sin t$ and $|t|$,

we get that $\lim_{t \to 0} \sin t = 0$

We shall use this result to calculate $\lim_{t \to 0} \cos t$. As you know, $\cos t = 1 - 2\sin^2 t/2$. This means

$$\lim_{t \to 0} \cos t = \lim_{t \to 0} (1 - 2\sin^2 t/2)$$

$$= 1 - 2 \lim_{t \to 0} \sin^2 t/2$$

$$= 1$$

We can prove that $\lim_{t \to 0} \sin^2 t/2 = 0$ by using Theorem 3 of Unit 2 and by noting that $t \to 0 \Leftrightarrow t/2 \to 0$

Thus, we get $\lim_{t \to 0} \cos t = 1$.

Now, let's get back to inequality (2): $\sin t < t < \tan t$ for $0 < t < \pi/2$. Since $0 < t < \pi/2$, $\sin t > 0$, and therefore, after dividing by $\sin t$, (2) becomes:

$$1 < t/\sin t < 1/\cos t$$

or $\cos t < \sin t/t < 1 \ldots (5), \; 0 < t < \pi/2$

Now, since $\sin(-t) = -\sin t$, we see that $\sin(-t)/(-t) = \sin t/t$. This, alongwith the result $\cos(-t) = \cos t$, shows that the inequality (5) holds even when $-\pi/2 < t < 0$. Thus,

$$\cos t < \sin t/t < 1, \quad -\pi/2 < t < \pi/2, \quad t \neq 0$$

Now, let us apply the sandwich theorem to the functions cos t, sin t/t and 1, and take the limits as $t \to 0$. This give us:

$$\lim_{t \to 0} \sin t/t = 1$$

Example 1. Suppose we want to find out

$$\lim_{t \to 0} \frac{\sin 3x}{x} \text{ and } \lim_{t \to 0} \frac{\sin 5x}{\sin 7x}$$

Let us first calculate $\lim\limits_{t \to 0} \frac{\sin 3x}{x}$. For this we shall write $\frac{\sin 3x}{x} = \frac{\sin 3x}{3x} \times 3$. If we replace 3x by t in the right hand side, and take the limit as $x \to 0$, we find that $t = 3x$ also tends to zero, and $\lim\limits_{x \to 0} \frac{\sin 3x}{x} = \lim\limits_{t \to 0} \frac{\sin t}{t} \times 3$

$$= 3 \lim_{t \to 0} \frac{\sin t}{t} \qquad \text{(See Theorem 3 of Unit 2)}$$

$$= 3$$

To calculate $\lim\limits_{x \to 0} \frac{\sin 5x}{\sin 7x}$ we start by writing

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 7x} = \lim_{x \to 0} \frac{\sin 5x}{5x} \times \frac{7x}{\sin 7x} \times \frac{5}{7}$$

$$= \frac{5}{7} \lim_{x \to 0} \frac{\sin 5x}{5x} \lim_{x \to 0} \frac{7x}{\sin 7x}$$

$$= \frac{5}{7} \left(\text{since } \lim_{x \to 0} \frac{7x}{\sin 7x} = \frac{1}{\lim\limits_{x \to 0} (\sin 7x/7x)} = 1 \text{ by Theorem 3 of Unit 2} \right)$$

Remark 1 In $\lim\limits_{t \to 0} \frac{\sin t}{t} = 1$ or $\lim\limits_{t \to 0} \cos t = 1$, the angle t is measured in radians. If in a particular problem, the angles are measured in degrees, we have to first convert these into radians before using these formulas. Thus,

$$\lim_{t \to 0} \frac{\sin t^\circ}{t} = \lim_{t \to 0} \frac{\sin (\pi t/180)}{t} = \frac{\pi}{180} \lim_{t \to 0} \frac{\sin (\pi t/180)}{\pi t/180} = \frac{\pi}{180}$$

See if you can solve this exercise now.

E E 1) Prove that a) $\lim\limits_{x \to 0} \cos(a + x) = \cos a$

b) $\lim\limits_{x \to 0} \sin(a + x) = \sin a$

4.2.2 Derivatives of Sin x and Cos x

We shall now find out the derivative of sinx from the first principles. If $y = f(x) = \sin x$, then by definition

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\sin(h/2)\cos(x + h/2)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(h/2)}{h/2} \lim_{h \to 0} \cos(x + h/2)$$

$$= 1 \times \cos x = \cos x$$

Remember the formula $\sin A - \sin B = 2\sin\frac{(A-B)}{2}\cos\frac{(A+B)}{2}$.

Thus, we get

$$\frac{d}{dx}(\sin x) = \cos x$$

Now, let us consider the function $y = f(x) = \cos x$ and find its derivative. In this case,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin(h/2)\sin(x + h/2)}{h}$$

$$= -\lim_{h \to 0} \frac{\sin(h/2)}{h/2} \lim_{h \to 0} \sin(x + h/2)$$

$$= -\sin x$$

Thus, we have shown that

$$\frac{d}{dx}(\cos x) = -\sin x$$

Actually, having first calculated $\frac{d}{dx}(\sin x)$, we could have found out the derivative of cos x by using the formula:

$\cos x = \sin(x + \pi/2)$. This gives us,

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}(\sin(x + \pi/2))$$

$$= \cos(x + \pi/2) = -\sin x$$

$\frac{d}{dx}(\sin(x + \pi/2)) = \cos(x + \pi/2)$ can be proved by using the chain rule.

In the next subsection we shall find the derivatives of the other four trigonometric functions by using similar formulas. But before that it is time to do some exercises.

E E 2) Find the derivatives of the following:

a) $\sin 2x$ b) $\cos^2 x$ c) $5\sin^7 x \sin 3x$ d) $x^3 \cos 9x$.

e) $\cos(\sin x)$.

4.2.3 Derivatives of other Trigonometric Functions

We shall now find the derivatives of i) tan x ii) cot x iii) sec x iv) cosec x.

i) Suppose $y = f(x) = \tan x$. We know that $\tan x = \frac{\sin x}{\cos x}$

Hence, $$\frac{dy}{dx} = \frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \; d/dx\,(\sin x) - \sin x \; d/dx\,(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\frac{d}{dx}(u/v) = \frac{v\,du/dx - u\,dv/dx}{v^2}$$

ii) Now, suppose $y = f(x) = \cot x$. Since $\cot x = 1/\tan x$, we get

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{\tan x \; d/dx\,(1) - 1 \; d/dx\,(\tan x)}{\tan^2 x}$$

$$= \frac{-\sec^2 x}{\tan^2 x} = -\operatorname{cosec}^2 x$$

iii) Now, let $y = f(x) = \sec x$. Since we know that $\sec x = 1/\cos x$, proceeding as in ii), we get

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos^2 x} = \sec x \; \tan x$$

If you have followed i), ii) and iii) above, you should not have any difficulty in finding the derivative of cosec x by using $\operatorname{cosec} x = 1/\sin x$.

E

E 3) Show that $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \; \cot x$.

Let us summarise our results.

Table 1

Function	Derivative
sin x	cos x
cos x	− sin x
tan x	$\sec^2 x$
cot x	$-\operatorname{cosec}^2 x$
sec x	sec x tan x
cosec x	− cosec x cot x

Remark 2 Here again we note that the angle is measured in radians. Thus,

$$\frac{d}{dx}(\sin x^\circ) = \frac{d}{dx}\left(\sin \frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi x}{180}\cos x^\circ$$

We shall now see how we can use these results to find the derivatives of some more complicated functions. The chain rule and the algebra of derivatives with which you must have become quite familiar by now, will come in handy again.

Example 2 Let us differentiate i) $\sec^3 x$ ii) $\sec x \tan x + \cot x$

i) Let $y = \sec^3 x$. If we write $u = \sec x$, we get $y = u^3$. Thus,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \sec x \tan x$$

$$= 3\sec^3 x \tan x$$

ii) If $y = \sec x \tan x + \cot x$, then,

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x \tan x) + \frac{d}{dx}(\cot x)$$

$$= \sec x \cdot \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) - \operatorname{cosec}^2 x$$

$$= \sec x(\sec^2 x + \tan^2 x) - \operatorname{cosec}^2 x$$

Remark 3 $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ are periodic functions with period 2π. Their derivatives are also periodic with period 2π. $\tan x$ and $\cot x$ are periodic with period π. Their derivatives are also periodic with period π.

We have been considering variables which are dimensionless. Actually, in practice, we may have to consider variables having dimensions of mass, length, time etc., and we have to be careful in interpreting their derivatives. Thus, we may be given that the distance x travelled by a particle in time t is $x = a \cos bt$. Here, since bt is dimensionless (being an angle), b must have the dimension $\frac{1}{T}$. Similarly, $x/a = \cos bt$ has to be dimensionless. This means that a must have the same dimension as x. That is dimension of a is L.

Now $dx/dt = -ab \sin bt$ has the dimension of $ab = L \times 1/T = L/T$, which is not unexpected, since dx/dt is nothing but the velocity of that particle.

See if you can do these exercises now.

E 4) Find the derivatives of:

a) $\operatorname{cosec} 2x$ b) $\cot x + \sqrt{\operatorname{cosec} x}$ c) $5\cot 9x$

4.3 DERIVATIVES OF INVERSE FUNCTIONS

We have seen in Unit 1 that the graphs of a function and its inverse are very closely related to each other. If we are given the graph of a function, we have only to take its reflection in the line $y = x$, to obtain the graph of its inverse. In this section we shall establish a relation between the derivatives of a function and its inverse.

4.3.1 The Inverse Function Theorem

Let us take two functions f and g, which are inverses of each other. We have already seen in Unit 1 that in this case, $gof(x) = g(f(x)) = x$, for all x for which f is defined, $fog(y) = f(g(y)) = y$ for all y for which g is defined. Now, suppose that both f and g are differentiable. Then, by applying the chain rule to differentiate $g(f(x)) = x$, we get $g'(f(x)).f'(x) = 1$ or $g'(y).f'(x) = 1$, where $y = f(x)$.

This means that if $f'(x) \neq 0$, we can write $g'(y) = 1/f'(x)$. So we have been able to find some relation between the derivatives of these inverse functions. Let us state our results more precisely.

The strict monotonicity condition in this theorem implies that f is one-one and thus ensures the existence of f^{-1}

Theorem 1 (The Inverse Function Theorem)
Let f be differentiable and strictly monotonic on an interval I. If $f'(x) \neq 0$ at a certain x in I, then f^{-1} is differentiable at $y = f(x)$ and
$(f^{-1})'(y) = 1/f'(x)$.

Thus, we have the inverse function rule:

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{dy/dx}$$

The derivative of the inverse function is the reciprocal of the derivative of the given function.

Soon we shall see that this rule is very useful if we want to find the derivative of a function when the derivative of its inverse function is already known. This will become clear when we consider the derivatives of the inverses of some standard functions. But first, let us use this rule to find the derivative of $f(x) = x^r$, where r is a rational number. In Unit 3 we have already proved that $\frac{d}{dx}(x^n) = nx^{n-1}$ when n is an integer. We shall use this fact in proving the general case.

Theorem 2 If $y = f(x) = x^r$, where r is a rational number for which x^r and x^{r-1} are both defined, then $\frac{d}{dx}(x^r) = rx^{r-1}$

x^r may not be always defined. For example, if $x = -1$ and $r = 1/2$, $x^r = \sqrt{-1}$ is not defined in R.

Proof: Let us first consider the case when $r = 1/q$, q being any non-zero integer. In this case, $y = f(x) = x^{1/q}$. Its inverse function g will be given by $x = g(y) = y^q$. This means

$$\frac{dx}{dy} = g'(y) = qy^{q-1}$$

Thus, by the inverse function rule, we get

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{qy^{q-1}}$$

$$= \frac{1}{q(x^{1/q})^{q-1}}$$

$$= \frac{1}{qx^{(q-1)/q}} = \frac{1}{q}x^{-(q-1)/q}$$

$$= \frac{1}{q}x^{(1/q)-1} = rx^{r-1}$$

So far, we have seen that the theorem is true when r is of the form 1/q, where q is an integer. Now, having proved this, let us take the general case when $r = p/q$, $p, q \in Z$(q is, of course, non-zero). Here,

$y = f(x) = x^r = x^{p/q}$

So, $\frac{dy}{dx} = \frac{d}{dx}(x^{p/q}) = \frac{d}{dx}(x^{1/q})^p$

Now, $\frac{d}{dx}(x^{1/q})^p = p(x^{1/q})^{p-1} \cdot \frac{d}{dx}(x^{1/q})$, by chain rule

$$= p(x^{1/q})^{p-1} \; (1/q)\, x^{(1/q)-1}$$
$$= (p/q)\, x^{(p/q)-1}$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}(x^r) = (p/q)x^{(p/q)-1} = rx^{r-1}$$

This completes the proof of the theorem.

The usefulness of this theorem is quite clear from the following example.

Example 3 Suppose we want to differentiate

$$y = (x^{5/6} + \sqrt{x})^{1/11}$$

We write $u = x^{5/6} + \sqrt{x}$. This gives us, $y = u^{1/11}$.

By chain rule, we get

$$\frac{dy}{dx} = \frac{1}{11}(x^{5/6} + \sqrt{x})^{(1/11)-1}\left(\frac{5}{6}x^{(5/6)-1} + \frac{1}{2}x^{-1/2}\right)$$

Thus,

$$\frac{dy}{dx} = \frac{1}{66}(x^{5/6} + \sqrt{x})^{-10/11}(5x^{-1/6} + 3x^{-1/2})$$

Why don't you try these exercises now?

E E 5) Differentiate

a) $5(x^3 + x^{1/3})$ b) $(\sqrt[5]{x} - \sqrt[9]{x})x^{\frac{1}{2}}$

4.4 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

In the last section we have seen how the inverse function theorem helps us in finding the derivative of x^n where n is a rational number. We shall now use that theorem to find the derivatives of inverse trigonometric functions.

We have noted in Unit 1, Section 5, that sometimes when a given function is not one-one, we can still talk about its inverse, provided we restrict its domain suitably. Now, sin x is neither a one-one, nor an onto function from R to R. But if we restrict its domain to $[-\pi/2, \pi/2]$, and co-domain to $[-1, 1]$, then it becomes a one-one and onto function, and hence the existence of its inverse is assured. In a similar manner we can talk about the inverses of the remaining trigonometric functions if we place suitable restrictions on their domains and co-domains.

Now that we are sure of the existence of inverse trigonometric functions, let's go ahead and find their derivatives.

4.4.1 Derivatives of $\sin^{-1} x$ and $\cos^{-1} x$

Let us consider the function $y = f(x) = \sin x$ on the domain $[-\pi/2, \pi/2]$. Fig. 3(a) shows the graph of this function. Its inverse is given by $g(y) = \sin^{-1}(y) = x$. We can see clearly that $\sin x$ is strictly increasing on $[-\pi/2, \pi 2]$.

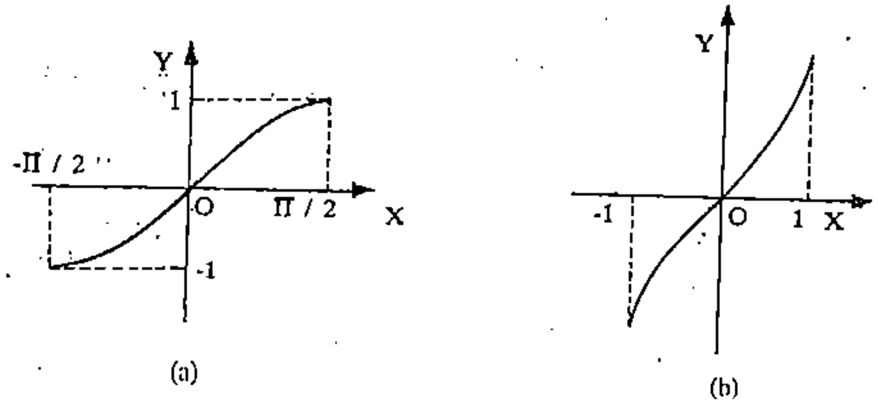


Fig.3

We also know that the derivative $\frac{d}{dx}(\sin x) = \cos x$ exists and is non-zero for all $x \in]-\pi/2, \pi/2[$.

This means that $\sin x$ satisfies the conditions of the inverse function theorem. We can, therefore, conclude that $\sin^{-1} y$ is differentiable on $]-1, 1[$, and

Since $\sin x = y$, $\cos x = \sqrt{1-y^2}$ for $-\pi/2 < x < \pi/2$.

$$\frac{d}{dy}(\sin^{-1}(y)) = \frac{1}{f'(x)} = \frac{1}{\cos x}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

Thus, we have the result

$$\frac{d}{dt}(\sin^{-1} t) = \frac{1}{\sqrt{1-t^2}}$$

Remember, $\sin^{-1} x$ is not the same as $(\sin x)^{-1} = 1/\sin x$ or $\sin x^{-1} = \sin 1/x$.

Fig.3(b) shows the graph of $\sin^{-1} x$.

We shall follow exactly the same steps to find out the derivative of the inverse cosine function.

Let's start with the function $y = f(x) = \cos x$, and restrict its domain to $[0, \pi]$ and its co-domain to $[-1,1]$. Its inverse function $g(y) = \cos^{-1} y$ exists and the graphs of $\cos x$ and $\cos^{-1} x$ are shown in Fig.4(a) and 4(b), respectively.

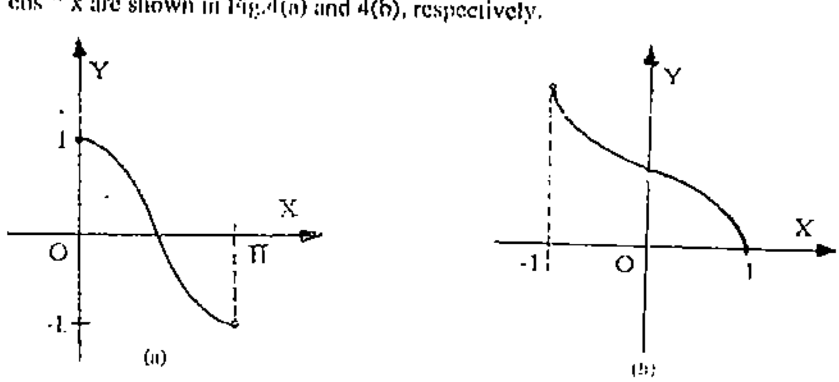


Fig.4

As in the earlier case, we can now check that the conditions of the inverse function theorem are satisfied and conclude that $\cos^{-1} y$ is differentiable in $]-1, 1[$. Further

$$\frac{d}{dy}(g(y)) = \frac{d}{dy}(\cos^{-1} y) = \frac{1}{f'(x)} = \frac{1}{-\sin x}$$

$$= \frac{-1}{\sqrt{1-y^2}}$$

Since $\cos x = y$, $\sin x = \sqrt{1-y^2}$ for $0 < x < \pi$.

This gives us the result.

$$\frac{d}{dt}(\cos^{-1} t) = \frac{-1}{\sqrt{1-t^2}}$$

You can apply these two results to get the derivatives in the following exercise.

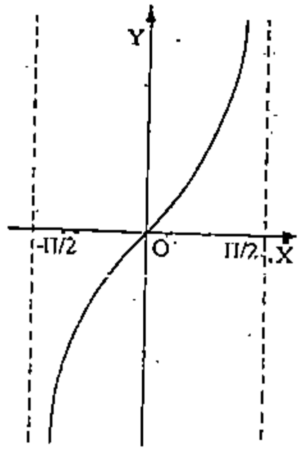
E E 6) Differentiate

a) $\sin^{-1}(5x)$ b) $\cos^{-1}\sqrt{x}$ c) $\sin x \cos^{-1}(x^3+2)$

E E 7) a) By looking at the graph of tan x given alongside, indicate the interval to which the domain of tan x should be restricted so that the existence of its inverse is guaranteed.

b) What will be the domain of $\tan^{-1} x$?

c) Prove that $\frac{d}{dx}(\tan^{-1} x) = 1/(1+x^2)$ in its domain.



In this section we have calculated the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ and if you have done E 7), you will have calculated the derivative of $\tan^{-1} x$ also. Proceeding along exactly similar lines, we shall be able to see that

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

4.4.2 Derivatives of $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$

Let's tackle the inverses of the remaining two trigonometric functions now.

To find $\sec^{-1} x$, we proceed as follows:

If $y = \sec^{-1} x$, then $\sec y = x$ or $1/\cos y = x$, which means that $1/x = \cos y$. This gives us

$y = \cos^{-1}(1/x)$, where, $|x| \geq 1$.

Thus, $y = \sec^{-1} x = \cos^{-1}(1/x), |x| \geq 1$

Remember, we have seen that $\cos^{-1}$ is defined in the interval $[-1, 1]$.

From this we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1}(1/x))$$

$$= \frac{-1}{\sqrt{1-1/x^2}} \cdot \frac{d}{dx}(1/x)$$

$$= \frac{-|x|}{\sqrt{x^2-1}}(-1/x^2)$$

$$= \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

Note that although $\sec^{-1}x$ is defined for $|x| \geq 1$, the derivative of $\sec^{-1} x$ does not exist when $x = 1$.

Thus, we have

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

E E 8) Following exactly similar steps, show that

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

Example 4 Suppose we want to find the derivative of $y = \sec^{-1} 2\sqrt{x}$.

By chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sec^{-1} 2\sqrt{x})$$

$$= \frac{1}{2\sqrt{x}\,\sqrt{4x-1}} \frac{d}{dx}(2\sqrt{x})$$

$$= \frac{1}{2\sqrt{x}\,\sqrt{4x-1}} \times \frac{1}{\sqrt{x}}$$

$$= \frac{1}{2x\sqrt{4x-1}}$$

Now, you will be able to solve these exercises using the results about the derivatives of inverse trigonometric functions.

E E 9) Differentiate,

a) $\cot^{-1}(x/2)$ b) $\dfrac{\cot^{-1}(x+1)}{\tan^{-1}(x+1)}$

c) $\cos^{-1}(5x+4)$ d) $\sec^{-1}\left(\dfrac{x\sin\theta}{1-x\cos\theta}\right)$

e) $\operatorname{cosec}^{-1}(x+1)+\sec^{-1}(x-1)$

4.5 USE OF TRANSFORMATIONS

Sometimes the process of finding derivatives is simplified to a large extent by making use of some suitable transformations. In this section we shall see some examples which will illustrate this fact.

Example 5 Suppose we want to find the derivative of

$y = \cos^{-1}(4x^3-3x)$

As you know, we can differentiate this function by using the formula for the derivative of $\cos^{-1} x$ and the chain rule. But suppose we put $x = \cos\theta$, then we get

$$y = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$$

$$= \cos^{-1}(\cos 3\theta) \qquad (\cos 3\theta = 4\cos^3\theta - 3\cos\theta)$$

$$= 3\theta$$

$= 3\cos^{-1} x$. Now this is a much simpler expression, and can be differentiated easily as:

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$$

Example 6 To differentiate $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, we use the transformation $x = \tan\theta$. This gives us,

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left[\frac{1-(1-2\sin^2\theta/2)}{2\sin\theta/2\,\cos\theta/2}\right]$$

$$= \tan^{-1}(\tan\theta/2)$$

$$= \theta/2 = \frac{\tan^{-1} x}{2}.$$

Now, we can write $\frac{dy}{dx} = \frac{1}{2(1+x^2)}$

Let's tackle another problem.

Example 7 Suppose we want to differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

For this, let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $z = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$. Our aim is to find dy/dz. We shall use the transformation $x = \tan\theta$. This gives us

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta, \text{ and}$$

$$z = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta.$$

Now if we differentiate y and z with respect to θ, we get $dy/d\theta = 2$ and $dz/d\theta = 2$.

Therefore, $\frac{dy}{dz} = \frac{dy/d\theta}{dz/d\theta} = 1$.

Alternatively, we have $y = z$. Hence, $dy/dz = 1$.

So, you see, a variety of complex problems can be solved easily by using transformations. The key to a successful solution is, however, the choice of a suitable transformation. We are giving some exercises below, which will give you the necessary practice in choosing the right transformation.

E E 10) Find the derivatives of the following functions using suitable transformations :

a) $\sin^{-1}(3x - 4x^3)$ b) $\cos^{-1}(1 - 2x^2)$

c) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ d) $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

e) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Now let us summarise the points covered in this unit.

4.6 SUMMARY

In this unit we have

1 calculated the derivatives of trigonometric functions:

Function	Derivative
sin x	cos x
cos x	−sin x
tan x	$\sec^2 x$
cot x	$-\operatorname{cosec}^2 x$
sec x	sec x tan x
cosec x	−cosec x cot x

2 discussed the inverse function theorem and

used the rule

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(y)}$$

to prove that $d/dx(x^r) = rx^{r-1}$, where r is a rational number.

3 used the inverse function theorem to find the derivatives of inverse trigonometric functions:

function	Derivative
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$, $-1 < x < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}$, $x \in R$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$, $x \in R$
$\sec^{-1} x$	$\frac{1}{\|x\|\sqrt{x^2-1}}$, $\|x\| > 1$
$\operatorname{cosec}^{-1} x$	$\frac{-1}{\|x\|\sqrt{x^2-1}}$, $\|x\| > 1$

4 used transformations to simplify the problems of finding the derivatives of some functions.

4.7 SOLUTIONS AND ANSWERS

E 1) a) $\cos(a + x) = \cos a \cos x - \sin a \sin x$

$\lim_{x \to 0} \cos(a + x) = \cos a \lim_{x \to 0} \cos x - \sin a \lim_{x \to 0} \sin x$

$= \cos a$

b) similar

E 2) a) $2 \cos 2x$ b) $2 \cos x \frac{d}{dx}(\cos x) = -2 \sin x \cos x$

c) $5(3\sin^7 x \cos 3x + 7\sin^6 x \cos x \sin 3x)$

$= 5 \sin^6 x (3 \sin x \cos 3x + 7\cos x \sin 3x)$

d) $3x^2 \cos 9x - 9x^3 \sin 9x$

e) $-\sin(\sin x) \cos x$

E 3) $\frac{d}{dx}(\operatorname{cosec} x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x \times 0 - 1 \cos x}{\sin^2 x}$

$= -\frac{\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x.$

E 4) a) $-2\operatorname{cosec} 2x \cot 2x$

b) $-\operatorname{cosec}^2 x + \frac{1}{\sqrt{2\operatorname{cosec} x}}(-\operatorname{cosec} x \cot x)$

c) $-45 \operatorname{cosec}^2 9x.$

E 5) a) $5(3x^2 + \frac{1}{3}x^{-2/3})$

b) $2(\sqrt[5]{x} - \sqrt[9]{x})x + x^2(\frac{1}{5}x^{-4/5} - \frac{1}{9}x^{-8/9})$

E 6) a) $\frac{5}{\sqrt{1-25x^2}}$ (b) $\frac{-1}{2\sqrt{x}\sqrt{1-x}}$

c) $\sin x \frac{-3x^2}{\sqrt{1-(x^3+2)^2}} + \cos x \cos^{-1}(x^3 + 2)$

E 7) a) $\tan x$ restricted to $]-\pi/2, \pi/2[$ is a strictly increasing one-one function of x. Thus, its inverse exists when restricted to $]-\pi/2, \pi/2[$.

b) The domain of $\tan^{-1} x$ is $]-\infty, \infty[$.

c) If $y = f(x) = \tan x$,

$$\frac{d}{dy}(\tan^{-1} y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1+y^2}$$

Hence $d/dx\ (\tan^{-1} x) = \frac{1}{1+x^2}$.

E 8) $y = \operatorname{cosec}^{-1} x \Rightarrow \operatorname{cosec} y = x \Rightarrow \sin y = 1/x \Rightarrow$

$y = \sin^{-1}(1/x)$ where $|x| \geq 1$.

Thus, $\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}(1/x))$

$$= \frac{1}{\sqrt{1-1/x^2}}(-1/x^2)$$

$$= \frac{|x|}{\sqrt{x^2-1}}(-1/x^2)$$

$$= \frac{-1}{|x|\sqrt{x^2-1}},\ |x| > 1$$

E 9) a) $\frac{-1}{2(1+x^2/4)}$

b) $$\frac{-\tan^{-1}(x+1)\left(\frac{1}{1+(x+1)^2}\right) - \cot^{-1}(x+1)\left(\frac{1}{1+(x+1)^2}\right)}{(\tan^{-1}(x+1))^2}$$

c) $\frac{5}{\sqrt{1-(5x+4)^2}}$

d) $$\frac{1}{\left|\frac{x\sin\theta}{1-x\cos\theta}\right|\sqrt{\frac{x^2\sin^2\theta}{(1-x\cos\theta)^2}-1}}\left[\frac{\sin\theta(1-x\cos\theta)+x\sin\theta\cos\theta}{(1-x\cos\theta)^2}\right]$$

e) $$\frac{-1}{|x+1|\sqrt{(x+1)^2-1}} + \frac{1}{|x-1|\sqrt{(x-1)^2-1}}$$

E 10) a) Put $x = \sin\theta \Rightarrow y = \sin^{-1}(3x-4x^3)$

$= \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x$.

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$$

b) $x = \cos\theta/2 \Rightarrow y = \cos^{-1}(1-2x^2)$

$\cos^{-1}(1-2\cos^2\theta/2)$

$\Rightarrow y = \cos^{-1}(-\cos\theta) = \cos^{-1}(\cos(\pi-\theta))$

$= \pi - \theta = \pi - 2\cos^{-1} x$.

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

c) Put $x = \tan\theta \Rightarrow y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1} x$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

d) Put $x = \tan\theta$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\right) = \frac{3}{1+x^2}$$

e) Put $x = \tan\theta$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right) = \frac{2}{1+x^2}$$

UNIT 5 DERIVATIVES OF SOME STANDARD FUNCTIONS

Structure

5.1 INTRODUCTION

Exponential functions occupy an important place in pure and applied science. Laws of growth and decay are very often expressed in terms of these functions. In this unit we shall study the derivatives of exponential functions. The inverse function theorem which we stated in Unit 4 will then help us to differentiate their inverses, the logarithmic functions. In particular, you will find that the natural exponential function is its own derivative.

Further, we shall introduce and differentiate hyperbolic functions and their inverses since they hold special significance for physical sciences. We shall demonstrate the method of finding derivatives by taking logarithms, and also that of differentiating implicit functions.

With this unit we come to the end of our quest for the derivatives of some standard, frequently used functions. In the next block we shall see the geometrical significance of derivatives and shall use them for sketching graphs of functions.

Objectives

After studying this unit you should be able to:

- find the derivatives of exponential and logarithmic functions
- define hyperbolic functions and discuss the existence of their inverses
- differentiate hyperbolic functions and inverse hyperbolic functions
- use the method of logarithmic differentiation for solving some problems
- differentiate implicit functions and also those functions which are defined with the help of a parameter.

5.2 EXPONENTIAL FUNCTIONS

Our main aim, here, is to find the derivatives of exponential functions. But let us first recall the definition of an exponential function.

5.2.1 Definition of an Exponential Function

A function f defined on R by $f(x) = a^x$, where $a > 0$, is known as an **exponential function**. Now to find the derivative of f, we shall have to take the limit:

$$\lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x \cdot \lim_{h\to 0}\frac{a^h-1}{h} \quad \text{.............. (1)}$$

So, if we put $\lim_{h\to 0}\frac{a^h-1}{h} = k$, we get

$\frac{d}{dx}a^x = k.a^x$. We can also interpret k as the derivative of a^x at $x = 0$. In Fig.1 you can see the graphs of exponential functions for various values of a.

a^x may not be defined for all x if $a < 0$. For example, $(-2)^{\frac{1}{2}}$ is not defined in R.

$$k = \lim_{h\to 0}\frac{a^h-1}{h}$$
$$= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$= f'(0)$$
$$= \frac{d}{dx}(a^x)\Big|_{x=0}$$

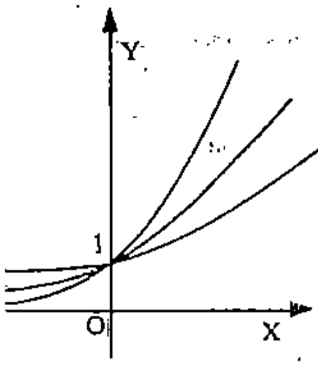


Fig.1

All these curves pass through (0, 1) as $a^0 = 1$ for all a. Now from all these curves, we shall choose that one, whose tangent at (0, 1) has slope = 1. (We assume that such a curve exists). The value of a corresponding to this curve is then denoted by e. Thus, we have singled out the exponential function : $x \to e^x$, so that its derivative at $x = 0$ is 1. Thus,

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1$$

This also means that

$$\frac{de^x}{dx} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x \cdot \lim_{h\to 0}\frac{e^h-1}{h} = e^x$$

That is, the derivative of this function is the function itself.

This special exponential function is called the **natural exponential function.**

5.2.2 Derivative of an Exponential Function

In Unit 1, we compared the graphs of the natural exponential function e^x and the natural logarithmic function lnx and found that they are reflections of each other w.r.t. the line $y = x$. (see Fig.2) We concluded that $= e^x$ and lnx are inverses of each other. This also means that $e^{\ln x} = x \ \forall x > 0$.

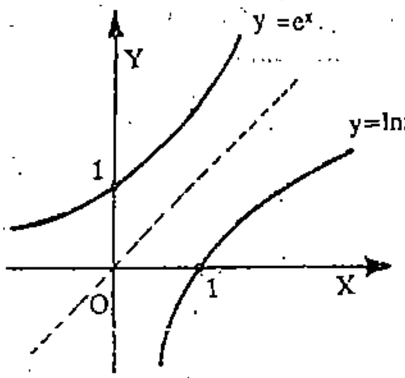


Fig.2

From this we can write $a^x = e^{\ln a^x}$, or $a^x = e^{x\ln a}$, where $a > 0$.

$\ln a^b = b\ln a$

$$\text{Thus } \frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a}$$
$$= e^{x\ln a}\frac{d}{dx}(x\ln a) \text{ by chain rule,}$$
$$= e^{x\ln a}\ln a$$
$$= a^x \ln a.$$

Remark 1 If we compare this result with (1) which we derived at the beginning of this section, we find that

$$\ln a = \lim_{h \to 0} \frac{a^h - 1}{h}$$

Thus, we have

$$\frac{d}{dx} e^x = e^x, \text{ and}$$

$$\frac{d}{dx} a^x = a^x \ln a$$

Example 1 Let us use these formulas to find the derivatives of

i) $e^{(x^2+2x)}$ ii) $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ iii) $a^{\sin^{-1} x}$

i) Let $y = e^{(x^2+2x)}$. Then, by chain rule

$$\frac{dy}{dx} = e^{(x^2+2x)} (2x + 2)$$

Hence $\frac{d}{dx}\left(e^{(x^2+2x)}\right) = 2(x+1)\, e^{(x^2+\ldots}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

ii) $$\frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right) = \frac{(e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x}) - (e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x})}{(e^x - e^{-x})^2}$$

$$= \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$$

$$= \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}$$

$$= \frac{-4}{(e^x - e^{-x})^2}$$

iii) We apply the chain rule again to differentiate $a^{\sin^{-1} x}$

$$\frac{d}{dx}\left(a^{\sin^{-1} x}\right) = \ln a \; a^{\sin^{-1} x} \frac{d}{dx}(\sin^{-1} x)$$

$$= \ln a \frac{1}{\sqrt{1 - x^2}} a^{\sin^{-1} x}$$

See if you can solve these exercises now.

E E 1) Find the derivatives of:

a) $5e^{(x^2-2)x}$ b) $e^{(x+1)/x}$ c) $(x+2)\, e^{\sqrt{x}}$ d) $e^{-m\tan^{-1} x}$ e) 2^{2x} f) $7^{\cos x}$

E 2) How much faster is $f(x) = 2^x$ increasing at $x = 1/2$ than at $x = 0$?

In this section we have defined e as that real number for which $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.
Alternatively, e can also be defined as a limit:

$e = \lim_{n \to \infty} (1 + 1/n)^n$, or as the sum of an infinite series: $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots$

But all these definitions give the same value, e = 2.718281828,... e is an irrational number.

In many situations the rate of growth (of human beings, or bacteria or radioactive particles) is proportional to the present population. That is, if x(t) is the population at time t, then $\frac{dx}{dt} \propto x$. In such situations the exponential function is of great relevance since $\frac{d}{dt}(e^t) = e^t$.

Now let us turn our attention to logarithmic functions.

5.3 DERIVATIVES OF LOGARITHMIC FUNCTIONS

In Unit 4, we studied the inverse function theorem, (Theorem 1, Unit 4) and used it to find the derivatives of various functions such as $\sin^{-1}x$, $\cos^{-1}x$, and so on. Here, we shall, yet again, apply this theorem to calculate the derivative of the natural logarithmic function.

5.3.1 Differentiating the Natural Log Function

Consider the function $y = \ln x$. This is the inverse of the natural exponential function, that is, $y = \ln x$ if and only if $x = e^y$.

From Fig. 2, you can see that the natural exponential function is a strictly increasing function. (You will be able to rigorously prove this result by the end of this course). Further, the derivative of the function $x = e^y$ is

lnx is defined on $]0, \infty[$.

$$\frac{dx}{dy} = \frac{d}{dy}(e^y) = e^y > 0 \text{ for all } y \in \mathbb{R}.$$

Thus, all the conditions of the inverse function theorem are satisfied. This means we can conclude that the derivative of the natural logarithmic function (which is the inverse of the natural exponential function) exists, and

$$\frac{dy}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}$$

Thus, we have

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Let's see how we can use this result.

Example 2 Suppose we want to differentiate $y = \ln(x^2 - 2x + 2)$.

$$\frac{dy}{dx} = \frac{1}{x^2 - 2x + 2}\frac{d}{dx}(x^2 - 2x + 2)$$
$$= \frac{2x - 2}{x^2 - 2x + 2}$$

Note that $x^2 - 2x + 2 = (x-1)^2 + 1$ and hence, is non-zero for all x.

Therefore, $\ln(x^2 - 2x + 2)$ is well-defined.

Example 3 If we want to differentiate $y = \ln\left|\frac{1+x^2}{1-x^2}\right|$, $|x| \neq 1$, we will have to consider two cases: i) $|x| > 1$ and ii) $|x| < 1$

i) If $|x| > 1$, we get $\left|\frac{1+x^2}{1-x^2}\right| = \frac{1+x^2}{-(1-x^2)} = \frac{x^2+1}{x^2-1}$,

since $|x| > 1$ makes $1-x^2$ negative. So in this case,

$$\frac{dy}{dx} = \frac{x^2-1}{x^2+1}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$
$$= \frac{4x}{1-x^4}, \text{ after simplification.}$$

ii) when $|x| < 1$, $\left|\frac{1+x^2}{1-x^2}\right| = \frac{1+x^2}{1-x^2}$ and so,

$$\frac{dy}{dx} = \frac{1-x^2}{1+x^2}\frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$$
$$= \frac{4x}{1-x^4}$$

So, we see that $\frac{dy}{dx} = \frac{4x}{1-x^4}$, for all x such that $|x| \neq 1$.

Now, let us turn our attention to logarithmic functions with arbitrary bases.

5.3.2 Differentiating the General Log Function

a^x is a constant function for $a = 1$. Hence, it does not have any inverse. The log functions are thus defined only for $a \neq 1$.

Let us consider any positive number $a \neq 1$. We say $\log_a x = y$ if and only if $x = a^y$. Obviously, the natural logarithmic function ln x can be written as $\log_e x$.

Further, we know that $\log_a x = \log_a e \cdot \ln x$. This rule gives a connection between the natural and general logarithmic functions. We shall use this relationship to find the derivative of $\log_a x$.

So, if $y = \log_a x = \log_a e \ln x$,

$$\frac{dy}{dx} = \log_a e\frac{d}{dx}\ln x = \log_a e\,\frac{1}{x}$$

Thus, we arrive at

$$\frac{d}{dx}(\log_a x) = \log_a e \cdot \frac{1}{x}$$

If we put a = e in this, we get our earlier result:

$$\frac{d}{dx}\ln x = \frac{1}{x}, \text{ since } \log_e e = 1$$

Example 4 Let us differentiate $y = \log_7 \tan^3 x$

$$\frac{dy}{dx} = \log_7 e\,\frac{1}{\tan^3 x}\frac{d}{dx}(\tan^3 x)$$
$$= \log_7 e\,\frac{1}{\tan^3 x}\,3\tan^2 x\,\sec^2 x$$
$$= 3\log_7 e\,\frac{\sec^2 x}{\tan x}$$

If you have followed the solved examples in this section you should have no difficulty in solving these exercises.

E E 3) Find the derivatives of:

a) $\log_2 2x$ b) $7\log_{11}(5x^2 + 2)$ c) $e^{-x}\ln x$

d) $\ln\left(\frac{1+x}{1-x}\right)$, $|x| < 1$ e) $\ln(\sin^4 x)$.

5.4 HYPERBOLIC FUNCTIONS

In applications of mathematics to other sciences, we, very often, come across certain combinations of e^x and e^{-x}. Because of their importance, these combinations are given special names, like the hyperbolic sine, the hyperbolic cosine etc. These names suggest that they have some similarity with the trigonometric functions. Let's look at their precise definitions and try to understand the points of similarity and dissimilarity between the hyperbolic and the trigonometric functions.

5.4.1 Definitions and Basic Properties

Definition 1 The hyperbolic sine function is defined by $\sinh x = \frac{e^x - e^{-x}}{2}$ for all $x \in R$. The range of this function is also R.

Definition 2 The hyperbolic cosine function is defined by $\cosh x = \frac{e^x + e^{-x}}{2}$ for all $x \in R$. The range of this function is $[1, \infty[$.

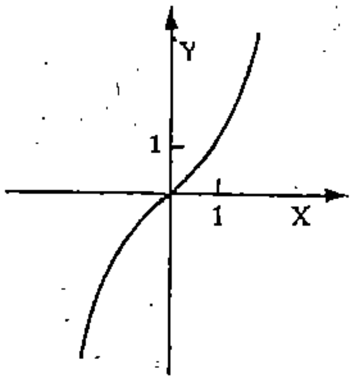
You will notice that

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = -\frac{e^x - e^{-x}}{2}$$

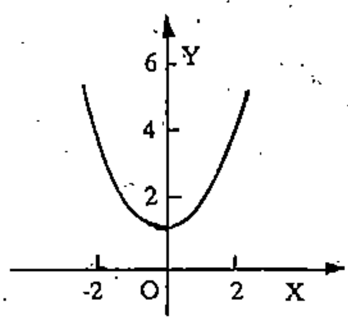
$$= -\sinh(x), \text{ and}$$

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

In other words, the hyperbolic sine is an odd function, while the hyperbolic cosine is an even function. Fig 3(a) and (b) show the graphs of these two functions.



(a)



(b)

Fig. 3 : Graph of (a) sinh x (b) cosh x

We also define four other hyperbolic functions as:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\text{sech } x = \frac{2}{e^x + e^{-x}}, \quad \text{cosech } x = \frac{2}{e^x - e^{-x}}.$$

E E 4) Verify that a) $\cosh^2 x - \sinh^2 x = 1$

b) $\tanh x = \dfrac{\sinh x}{\cosh x}$

c) $1 - \tanh^2 x = \text{sech}^2 x$.

E E 5) Derive an identity connecting coth x and cosech x.

You must have noticed that the identities involving these hyperbolic functions are similar to those involving trigonometric functions. It is possible to extend this analogy and get some more formulas:

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

Since we have seen that $\cosh^2 t - \sinh^2 t = 1$, it is obvious that a point with coordinates $(\cosh t, \sinh t)$ lies on the unit hyperbola: $x^2 - y^2 = 1$. (Hence the name, hyperbolic functions). We have a similar situation in the case of trigonometric functions. The point $(\cos t, \sin t)$ lies on the unit circle: $x^2 + y^2 = 1$. That is why trigonometric functions are also called circular functions.

There is one major point of difference between the hyperbolic and circular functions, though. While t in sint, cost, etc. is the measure of an angle, the t which appears in sinht, cosht, etc. cannot be interpreted as the measure of an angle. However, it is sometimes called the hypberbolic radian.

5.4.2 Derivatives of Hyperbolic Functions

Since the hyperbolic functions are defined in terms of the natural exponential function, whose derivative we already know, it is very easy to calculate their derivatives. For example,

$\sinh x = \dfrac{e^x - e^{-x}}{2}$. This means,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

Similarly, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ gives us

$$\frac{d}{dx}(\cosh x) = \frac{e^x - e^{-x}}{2} = \sinh x$$

In the case of $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, we get

$$\frac{d}{dx}(\tanh x) = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= 1 - \tanh^2 x = \text{sech}^2 x$$

We can adopt the same method for finding the derivatives of coth x, sech x and cosech x. In Table 1 we have collected all these results.

Table 1

Function	Derivative
sinh x	cosh x
cosh x	sinh x
tanh x	$\text{sech}^2 x$
coth x	$-\text{cosech}^2 x$
sech x	$-$ sech x tanh x
cosech x	$-$ cosech x coth x

Example 5 Suppose we want to find dy/dx when $y = \tanh(1 - x^2)$.

$$\frac{dy}{dx} = \text{sech}^2(1 - x^2) \cdot \frac{d}{dx}(1 - x^2)$$

$$= -2x\ \text{sech}^2(1 - x^2)$$

See if you can solve these exercises on your own.

E E 6) Find $f'(x)$ when $f(x) =$

a) $\tanh \frac{4x + 1}{5}$ b) $\sinh e^{2x}$ c) $\coth(1/x)$

d) $\text{sech}(\ln x)$ e) $e^x \cosh x$

5.4.3 Derivatives of Inverse Hyperbolic Functions

We shall try to find the derivatives of the inverse hyperbolic functions now. Let us start with the inverse hyperbolic sine function.

From Fig. 3(a) you can see that the hyperbolic sine is a strictly increasing function. This means that its inverse exists, and

$$y = \sinh^{-1} x \Leftrightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\Leftrightarrow 2x = e^y - e^{-y}$$

$$\Leftrightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Leftrightarrow (e^y)^2 - 2xe^y - 1 = 0$$

$$\Leftrightarrow e^y = x + \left(\sqrt{1 + x^2}\right)$$

$$\Leftrightarrow y = \ln\left(x + \sqrt{1 + x^2}\right)$$

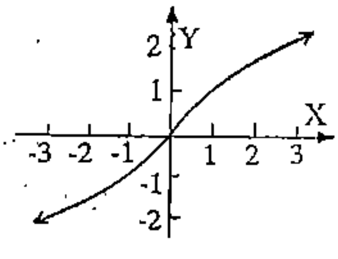


Fig. 4

We have used the formula for finding the roots of a quadratic equation here. Note that if $e^y = x - \sqrt{1 + x^2}$, then $e^y < 0$, which is impossible. Therefore we ignore this root.

Thus $\sinh^{-1}x = \ln(x + \sqrt{1 + x^2})$, $x \in]-\infty, \infty[$. In Fig. 4, we have drawn the graph of $\sinh^{-1}x$. Now,

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{d}{dx}\left(\ln\left(x + \sqrt{1 + x^2}\right)\right)$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \frac{d}{dx}\left(x + \sqrt{1 + x^2}\right)$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \left(1 + \frac{x}{\sqrt{1 + x^2}}\right)$$

$$= \frac{1}{\sqrt{1 + x^2}}$$

Thus, $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}} = \frac{1}{\sqrt{x^2 + 1}}$

In the case of the hyperbolic cosine function, we see from Fig. 3(b), that its inverse will exist if we restrict its domain to $[0, \infty[$. The domain of this inverse function will be $[1, \infty[$, and its range will be $[0, \infty[$.

Now $y = \cosh^{-1} x \Leftrightarrow x = \cosh y = \frac{e^y + e^{-y}}{2}$

$$\Leftrightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Leftrightarrow e^y = x + \sqrt{x^2 - 1}$$

$$\Leftrightarrow y = \ln\left(x + \sqrt{x^2 - 1}\right)$$

Thus $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$, $x \geq 1$.

Fig. 5 shows the graph of $\cosh^{-1} x$.

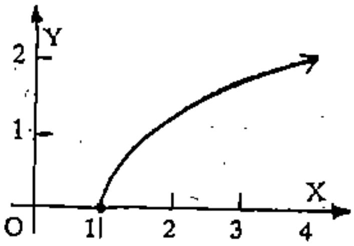


Fig. 5

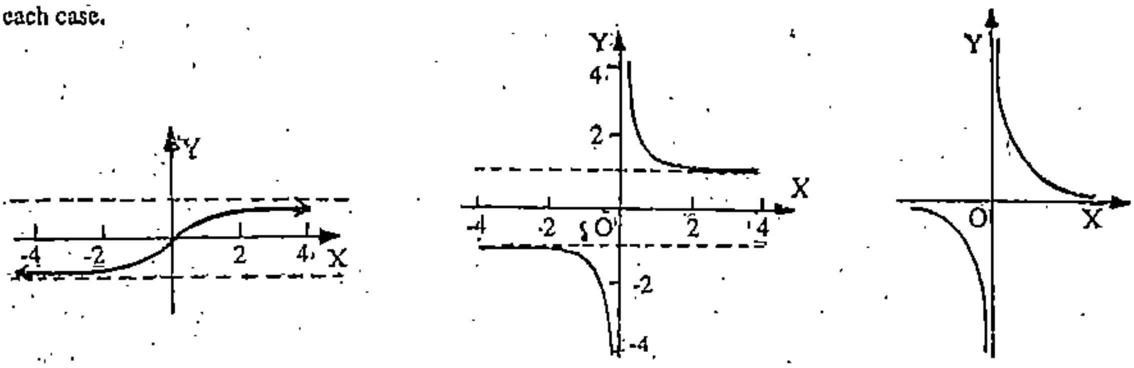
Again we ignore the root $e^y = x - \sqrt{x^2 - 1}$, because then $e^y < 1$, which is impossible since $y > 0$.

Further $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{x + \sqrt{x^2 - 1}} \frac{d}{dx}\left(x + \sqrt{x^2 - 1}\right)$

$$= \frac{1}{\sqrt{x^2 - 1}}, x > 1$$

Note that the derivative of $\cosh^{-1}x$ does not exist at $x = 1$.

Fig. 6 (a), (b) and (c) show the graphs of tanh x, coth x and cosech x. You can see that each of these functions is one-one and strictly monotonic. Thus, we can talk about the inverse in each case.



(a) (b) (c)

Fig. 6

Arguing as for $\sinh^{-1} x$ and $\cosh^{-1} x$, we get

$y = \tanh^{-1} x \Leftrightarrow x = \tanh y \Leftrightarrow y = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$, $|x| < 1$

$y = \coth^{-1} x \Leftrightarrow x = \coth y \Leftrightarrow y = \frac{1}{2}\ln\left(\frac{x - 1}{x + 1}\right)$, $|x| > 1$

$y = \operatorname{cosech}^{-1} x \Leftrightarrow x = \operatorname{cosech} y \Leftrightarrow y = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right)$, $x \neq 0$

Since $\operatorname{sech} x = \frac{1}{\cosh x}$, we shall have to restrict the domain of sech x to $[0, \infty[$ before talking about its inverse, as we did for cosh x. $\operatorname{Sech}^{-1}x$ is defined for all $x \in]0, 1]$, and we can write

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), \quad 0 < x \leq 1$$

Now, we can find the derivatives of each of these inverse hyperbolic functions. We proceed exactly as we did for the inverse hyperbolic sine and cosine functions and get

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}; \ |x|<1$$

$$\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}, \ |x|>1$$

$$\frac{d}{dx}(\text{sech}^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}, \ 0<x<1$$

$$\frac{d}{dx}(\text{cosech}^{-1}x) = \frac{-1}{|x|\sqrt{1+x^2}}, \ x \neq 0$$

Let us use these results to solve some problems now.

Example 6 Suppose we want to find the derivatives of a) $f(x) = \sinh^{-1}(\tan x)$, and b) $g(x) = \tanh^{-1}(\cos e^x)$.

Let's start with $f(x) = \sinh^{-1}(\tan x)$.

$$f'(x) = \frac{1}{\sqrt{\tan^2 x + 1}} \frac{d}{dx}(\tan x)$$

$$= \frac{1}{|\sec x|} \sec^2 x = |\sec x|$$

Now if $g(x) = \tanh^{-1}(\cos e^x)$, this means that

$$g'(x) = \frac{1}{1-\cos^2 e^x} \frac{d}{dx}(\cos e^x)$$

$$= \frac{1}{\sin^2 e^x}(-\sin e^x) \cdot e^x$$

$$= \frac{-e^x}{\sin e^x} = -e^x \text{cosec } e^x$$

We are now listing some functions for you to differentiate.

E E 7) Differentiate the following functions on their respective domains.

a) $\text{cosech}^{-1}(5\sqrt{x})$ b) $[\text{sech}^{-1}(\cos^2 x)]^{1/3}$

c) $\coth^{-1}(e^{(x^2+5x-6)})$

d) $\tanh^{-1}(\coth x) + \coth^{-1}(2x)$

e) $\sinh^{-1}\sqrt{x} + \cosh^{-1}(2x^2)$

5.5 METHODS OF DIFFERENTIATION

In this section, we shall study different methods of finding derivatives. We shall also see that the problem of differentiating some functions is greatly simplified by using these methods. Some of the results we derived in the earlier sections will be useful to us here.

5.5.1 Derivative of x^r

In Unit 4 we have seen that $\frac{d}{dx}(x^r) = rx^{r-1}$ when r is a rational number. Now, we are in a position to extend this result to the case when r is any real number. So if $y = x^r$, where $x > 0$ and $r \in \mathbf{R}$, we can write this as

$y = e^{\ln x^r} = e^{r\ln x}$, since the natural exponential and logarithmic functions are inverses of each other.

As we have mentioned in Unit 4, if $x < 0$, x^r may not be a real number. For example, $-3^{1/2} = \sqrt{-3} \notin \mathbf{R}$.

Thus $\frac{dy}{dx} = \frac{d}{dx}(e^{r\ln x}) = e^{r\ln x}\frac{d}{dx}(r\ln x)$

$= re^{r\ln x}\frac{1}{x} = \frac{rx^r}{x} = rx^{r-1}$

This proves that

$\frac{d}{dx}(x^r) = rx^{r-1}$ for $x > 0$, $r \in \mathbf{R}$.

We are sure, you will be able to solve this exercise now.

E 8) Differentiate,

a) $x^{\sqrt{2}}$ b) x^e

5.5.2 Logarithmic Differentiation

Sometimes we find that the process of taking derivatives becomes simple if we take logarithms before differentiating. In this section we shall illustrate this point through some

By $\sqrt{a}$ we mean the positive square root of a.

examples. But to take the logarithm of any quantity we have to be sure that it is non-negative. To overcome this difficulty, let us first try to find the derivative of ln (| x |).

Now you can check easily that $|x| = \sqrt{x^2}$.

Therefore, $\ln(|x|) = \ln\sqrt{x^2}$, and

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln\sqrt{x^2} = \frac{1}{\sqrt{x^2}}\frac{d}{dx}(\sqrt{x^2})$$

$$= \frac{1}{\sqrt{x^2}}\cdot\frac{x}{\sqrt{x^2}} = \frac{x}{x^2} = \frac{1}{x}$$

We get,

$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}$$

Using chain rule we can now conclude that if u is any function of x, then $\frac{d}{dx}\ln(|u|) = \frac{1}{u}\cdot\frac{du}{dx}$

Let us see how this result helps us in simplifying the differentiation of some functions.

Example 7 To differentiate $\dfrac{(x^2+1)^9(x-3)^{3/4}}{(x-5)^{2/3}(x^2+2x+1)^{-1/3}}$

we start by taking $y = \dfrac{(x^2+1)^9(x-3)^{3/4}}{(x-5)^{2/3}(x^2+2x+1)^{-1/3}}$

Thus, $|y| = \dfrac{|x^2+1|^9|x-3|^{3/4}}{|x-5|^{2/3}|x^2+2x+1|^{-1/3}}$

Then taking logarithms of both sides, we get

$$\ln|y| = \ln(|x^2+1|^9|x-3|^{3/4}) - \ln(|x-5|^{2/3}|x^2+2x+1|^{-1/3})$$

$$= \ln(|x^2+1|^9) + \ln(|x-3|^{3/4}) - \ln(|x-5|^{2/3}) - \ln(|x^2+2x+1|^{-1/3})$$

$$= 9\ln|x^2+1| + \frac{3}{4}\ln|x-3| - \frac{2}{3}\ln|x-5| + \frac{1}{3}\ln|x^2+2x+1|$$

ln (a/b) = lna − lnb
ln(ab) = lna + lnb
ln(a^b) = blna

Differentiating throughout we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{9}{x^2+1}2x + \frac{3}{4(x-3)} - \frac{2}{3(x-5)} + \frac{2x+2}{3(x^2+2x+1)}$$

$$= \frac{18x}{x^2+1} + \frac{3}{4(x-3)} - \frac{2}{3(x-5)} + \frac{2(x+1)}{3(x+1)^2}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{18x}{x^2+1} + \frac{3}{4(x-3)} - \frac{2}{3(x-5)} + \frac{2}{3(x+1)}\right]$$

$$= \frac{(x^2+1)^9(x-3)^{3/4}}{(x-5)^{2/3}(x^2+2x+1)^{-1/3}}\left[\frac{18x}{x^2+1} + \frac{3}{4(x-3)} - \frac{2}{3(x-5)} + \frac{2}{3(x+1)}\right]$$

Example 8 Suppose we want to differentiate $x^{\sin x}$, $x > 0$.

Let us write $y = x^{\sin x}$. Then $y > 0$ and so we can take logarithms of both sides to the base e and write

$\ln y = \ln x^{\sin x} = \sin x.\ln x$

Differentiating throughout, we get,

$$\frac{1}{y}\frac{dy}{dx} = \sin x\,\frac{1}{x} + \cos x\,\ln x$$

$$= \frac{\sin x}{x} + \cos x\ln x$$

Therefore $\dfrac{dy}{dx} = y\left(\dfrac{\sin x}{x} + \cos x\ \ln x\right)$

or $\dfrac{dy}{dx} = x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x\ \ln x\right)$

Example 9 To differentiate $x^{\cos x} + (\cos x)^x$, let $f(x) = x^{\cos x}$ and $g(x) = \cos x^x$. To ensure that $f(x)$ and $g(x)$ are well defined, let us restrict their domain to $[0, \pi/2]$.

$y = x^{\cos x} + (\cos x)^x = f(x) + g(x) > 0$ for $x \in [0, \pi/2]$

Let us differentiate both $f(x)$ and $g(x)$ by taking logarithms. We have,

$f(x) = x^{\cos x}$

Therefore $\ln f(x) = \cos x \ln x$.

Thus $\frac{1}{f(x)} f'(x) = -\sin x \ln x + \cos x \frac{1}{x}$.

That is, $f'(x) = f(x)\left(-\sin x \ln x + \frac{\cos x}{x}\right)$

$$= x^{\cos x}\left(\frac{-x\sin x \ln x + \cos x}{x}\right)$$

$$= x^{\cos x - 1}(\cos x - x\sin x \ln x)$$

Similarly, $g(x) = (\cos x)^x$ and so $\ln g(x) = x \ln \cos x$

Then $\frac{1}{g(x)} g'(x) = \ln \cos x + \frac{x}{\cos x}(-\sin x)$

$$\Rightarrow g'(x) = (\cos x)^x\left(\frac{\cos x \ln \cos x - x\sin x}{\cos x}\right)$$

$$= (\cos x)^{x-1}(\cos x \ln \cos x - x\sin x)$$

Hence, $\frac{dy}{dx} = f'(x) + g'(x)$

$$= x^{\cos x - 1}(\cos x - x\sin x \ln x) + \cos x^{x-1}(\cos x \ln \cos x - x\sin x)$$

If you have followed these examples you should have no difficulty in solving these exercises by the same method.

E E9) Differentiate,

a) $(x^2 - 1)(x^2 + 2)^6(x^3 - 1)^5$

b) $\frac{1}{(x-1)^5(x-2)^6(x-3)^7}$

c) $(\sin x)^x + (\cos x)^{\tan x}$

d) $(x^x)^x + x^{(x^x)}$

e) $(\sin x)^{\ln x} + x^x$

5.5.3 Derivatives of Functions Defined in Terms of a Parameter

Till now we were concerned with functions which were expressed as $y = f(x)$. We called x an independent variable, and y, a dependent one. But sometimes the relationship between two variables x and y may be expressed in terms of another variable, say t. That is, we may have a pair of equations $x = \phi(t)$, $y = \psi(t)$, where the functions ϕ and ψ have a common domain. For example, we know that the circle $x^2+y^2 = a^2$ is also described by the pair of equations, $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$.

In such cases the auxiliary variable t is called a parameter and the equations $x = \phi(t)$, $y = \psi(t)$ are called parametric equations. Now, suppose a function is defined in terms of a parameter. To obtain its derivative, we need only differentiate the relations in x and y separately. The following example illustrates this method.

Example 10 Let us try to find $\frac{dy}{dx}$ if $x = a\cos\theta$ and $y = b\sin\theta$.
(Here the parameter is θ)
We differentiate the given equations w.r.t. θ, and get

$$\frac{dy}{d\theta} = b\cos\theta, \quad \text{and} \quad \frac{dx}{d\theta} = -a\sin\theta$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta.$$

Try to apply this method now.

E E 10) Find $\frac{dy}{dx}$ if

a) $x = a\cos\theta$, $y = a\sin\theta$

b) $x = at^2$, $y = 2at$

c) $x = a\cos^3\theta$, $y = b\sin^3\theta$

d) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

5.5.3 Derivatives of Implicit Functions

It is not always necessary to express y explicitly in terms of x(as in $y = f(x)$) to find its derivative. We shall now see how to differentiate a function defined implicitly by a relation in x and y (such as, $g(x, y) = 0$).

Example 11 Let us find $\frac{dy}{dx}$ if x and y are related by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

Differentiating throughout with respect to x, we get

$$2ax + 2h \cdot 1 \cdot y + 2hx \cdot \frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

or $\frac{dy}{dx}(2hx + 2by + 2f) = -2ax - 2hy - 2g$

or $\frac{dy}{dx} = \frac{-(ax + hy + g)}{(hx + by + f)}$

See if you can find $\frac{dy}{dx}$ for the following implicit functions.

E E 11) Find $\frac{dy}{dx}$ if x and y are related as follows:

a) $x^2 + y^2 = 1$

b) $y^2 = 4ax$

c) $x^3y^3 + x^2y^2 + xy + 1 = 0$

d) $\cos x \cos y - y^2\sin^{-1}x + 2x^2 \tan x = 0$

5.6 SUMMARY

In this unit we have

1 obtained derivatives of the exponential and logarithmic functions, hyperbolic functions and their inverses. We give them in the following table.

Function	Derivative
e^x	e^x
$\ln x$	$\frac{1}{x}$
a^x	$a^x \ln a$
$\log_a x$	$\frac{1}{x} \log_a e$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$

Function	Derivative
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$, $x > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$, $\lvert x\rvert < 1$
$\coth^{-1} x$	$\frac{1}{1-x^2}$, $\lvert x\rvert > 1$
$\text{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}$, $0 < x < 1$
$\text{cosech}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$, $x \neq 0$.

2 extended the result $\frac{d}{dx}(x^r) = rx^{r-1}$ to all $r \in R$ and $x > 0$

3 illustrated

logarithmic differentiation,

differentiation of functions involving parameters and

differentiation of functions given by implicit relations

5.7 SOLUTIONS AND ANSWERS

E 1) a) $5e^{(x^2-2)}(2x)$ b) $e^{(x+1)/x}(-1/x^2)$

c) $(x+2)e^{\sqrt{x}}\ \frac{1}{2}x^{-1/2}+e^{\sqrt{x}}$

d) $e^{-m\tan^{-1}x}\left(\frac{-m}{1+x^2}\right)$

e) $2^{2x+1}\ln 2$

f) $7^{\cos x}(-\sin x)\ln 7$

E 2) $f'(x)=2^x\ln 2 \quad f'(0)=\ln 2$

$f'(1/2)=2^{1/2}\ln 2=\sqrt{2}\ln 2$

Hence f increases $\sqrt{2}$ times faster at $x = 1/2$ than at $x = 0$.

E 3) a) $\frac{1}{x}\log_2 e$

b) $7\log_{11}e\left(\frac{10x}{5x^2+2}\right)$

c) $e^{-x}(1/x)-e^{-x}\ln x$

d) $\frac{1-x}{1+x}\left[\frac{(1-x)+(1+x)}{(1-x)^2}\right]$

$=\frac{2}{(1-x)(1+x)}$

e) $\frac{1}{\sin^4 x}4\sin^3 x\ \cos x$

E 4) a) $\cosh^2 x=\frac{e^{2x}+e^{-2x}+2}{4}$, $\sinh^2 x=\frac{e^{2x}+e^{-2x}-2}{4}$

$\cosh^2 x-\sinh^2 x=1$

E 5) $\coth^2 x-1=\frac{e^{2x}+e^{-2x}+2}{e^{2x}+e^{-2x}-2}-1=\frac{4}{e^{2x}+e^{-2x}-2}=\text{cosech}^2 x$

E 6) a) $\frac{4}{5}\text{sech}^2\left[\frac{4x+1}{5}\right]$ b) $2e^{2x}\cosh e^{2x}$

c) $\frac{1}{x^2}\text{cosech}^2\left(\frac{1}{x}\right)$ d) $-\text{sech}\ln x\ \tanh\ln x\cdot\frac{1}{x}$

e) $e^x(\sinh x+\cosh x)$

E 7) a) $\frac{-1}{5\sqrt{x}\sqrt{1+25x}}\left(\frac{5}{2\sqrt{x}}\right)=\frac{-1}{2x\sqrt{1+25x}}$

b) $\frac{1}{3}[\text{sech}^{-1}(\cos^2 x)]^{-2/3}\left(\frac{1}{\cos^2 x\sqrt{1-\cos^4 x}}\right)\cdot 2\cos x\ \sin x$

c) $\frac{-1}{e^{2(x^2+5x-6)}-1}e^{(x^2+5x-6)}(2x+5)$

d) $\frac{-\text{cosech}^2 x}{1-\coth^2 x}-\frac{2}{4x^2-1}$

e) $\frac{1}{2\sqrt{x}\sqrt{1+x}}+\frac{4x}{\sqrt{4x^4-1}}$

E 8) a) $\sqrt{2}x^{\sqrt{2}-1}$ b) ex^{e-1}

E 9) a) $\ln|y|=\ln|x^2-1|+6\ln|x^2+2|+5\ln|x^3-1|$

$\frac{1}{y}\frac{dy}{dx}=\frac{2x}{x^2-1}+\frac{12x}{x^2+2}+\frac{15x^2}{x^3-1}$

$\frac{dy}{dx}=(x^2-1)(x^2+2)^6(x^3-1)^5\left[\frac{2x}{x^2-1}+\frac{12x}{x^2+2}+\frac{15x^2}{x^3-1}\right]$

b) $\ln|y|=-5\ln|x-1|-6\ln|x-2|-7\ln|x-3|$

$\frac{dy}{dx}=\frac{-1}{(x-1)^5(x-2)^6(x-3)^7}\left(\frac{5}{x-1}+\frac{6}{x-2}+\frac{7}{x-3}\right)$

c) Let $f(x) = (\sin x)^x$ and $g(x) = (\cos x)^{\tan x}$

Then $f'(x) = \sin x^x (\ln \sin x + x \cot x)$ and $g'(x) = \cos x^{\tan x} (\sec^2 x \ln \cos x - \tan^2 x)$

$\frac{dy}{dx} = f'(x) + g'(x)$

d) Let $f(x) = (x^x)^x$, $g(x) = x^{(x^x)}$, $x > 0$

If $y = x^x$, $\ln y = x \ln x$

$\Rightarrow \frac{dy}{dx} = x^x(1 + \ln x)$

$\ln f(x) = x \ln x^x$

$\Rightarrow \frac{1}{f(x)} f'(x) = \ln x^x + x(1 + \ln x)$

$\Rightarrow f'(x) = (x^x)^x [\ln x^x + x(1 + \ln x)]$

$\ln g(x) = x^x \ln x$

$\Rightarrow \frac{1}{g(x)} g'(x) = \frac{x^x}{x} + \ln x \; x^x(1 + \ln x)$

$\Rightarrow g'(x) = x^{(x^x)} [x^{x-1} + x^x \ln x (1 + \ln x)]$

Answer $= f'(x) + g'(x)$

$= (x^x)^x [\ln x^x + x(1 + \ln x)] + x^{(x^x)} [x^{x-1} + x^x \ln x(1 + \ln x)]$

e) $\frac{d}{dx} (\sin x)^{\ln x} = (\sin x)^{\ln x} \left(\ln x \cot x + \frac{\ln \sin x}{x} \right)$

$\frac{d}{dx} (x^x) = x^x (1 + \ln x)$

Answer $= (\sin x)^{\ln x} \left\{ \ln x \cot x + \frac{\ln \sin x}{x} \right\} + x^x (1 + \ln x)$

E 10) a) $\frac{dx}{d\theta} = -a \sin \theta$, $\frac{dy}{d\theta} = a \cos \theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\cot \theta$

b) $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

c) $\frac{dy}{dx} = \frac{3b \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\frac{b}{a} \tan \theta$

d) $\frac{dy}{dx} = \frac{a \sin \theta}{a(1 - \cos \theta)} = \frac{\sin \theta}{(1 - \cos \theta)}$

E 11 a) $2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

b) $2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

c) $3x^3y^2 \frac{dy}{dx} + 3x^2y^3 + 2x^2y \frac{dy}{dx} + 2xy^2 + x \frac{dy}{dx} + y = 0$

$\Rightarrow (3x^3y^2 + 2x^2y + x) \frac{dy}{dx} = -(3x^2y^3 + 2xy^2 + y)$

$\Rightarrow \frac{dy}{dx} = -\frac{(3x^2y^3 + 2xy^2 + y)}{(3x^3y^2 + 2x^2y + x)}$

d) [illegible]

$\Rightarrow \frac{dy}{dx} = \frac{\sin x \cos x \; [illegible] \; \frac{y^2}{\sqrt{(1-x^2)}} \; [illegible] \; \sec^2 x}{\cos x \sin y \; [illegible]}$

UTTAR PRADESH
RAJARSHI TANDON OPEN UNIVERSITY

UGMM - 01
Calculus

Block

2

DRAWING CURVES

BLOCK 2 DRAWING CURVES

In Block 1 you have learnt some techniques of differentiation, and have differentiated a wide variety of functions. In this block we shall use the derivative to explore various geometrical features of a curve, like concavity/convexity, maxima/minima, tangents, normals and so on. For this we will need to make use of not only the first derivative, but also some higher order derivatives.

Unit 6, the first unit of this block will introduce you to higher order derivatives. In the next two units, Units 7 and 8, we shall illustrate how we can find the exact shape of a curve, when its equation is given to us. You will be surprised at the amount of information which is revealed by the first, second and third order derivatives. We shall use this information to trace various standard curves in Unit 9. In Unit 9, we shall also tell you how the properties of certain remarkable curves are put to use. We shall also ask you to trace some curves yourself. Do try and trace them by systematically following the procedure which we have outlined in Unit 9. We are sure, that after reading this block you will be aware of the presence of many of these curves in the objects around you, as also in nature.

We have also made a video programme, "Curves", which you can see after going through this block. This programme is available at your study centre.

Also, after studying this block, you should attempt Assignment 1 of the course. Please submit your solutions to the assignment to your study centre coordinator, and ask her/him for the evaluated assignment after a month.

NOTATIONS AND SYMBOLS

$\frac{dy}{dx}$, y_1, $f'(x)$	the first derivative of y or f(x) w.r.t. x
$\frac{d^2y}{dx^2}$, y_2, $f''(x)$	the second derivative of y or f(x) w.r.t. x
$\frac{d^ny}{dx^n}$, y_n, $f^{(n)}(x)$	the nth derivative of y or f(x) w.r.t. x
$\simeq$	is approximately equal to

Also see the list of notations and symbols in Block 1.

UNIT 6 HIGHER ORDER DERIVATIVES

Structure

6.1 INTRODUCTION

In the first block you have differentiated a number of functions. You know that the derivative f' of a differentiable function f is again a function and is called the derived function of f.

We have already seen in Unit 3 that the concept of differentiation was motivated by some physical concepts (like the velocity of a moving particle) and also by geometrical notions (like the slope of a tangent to a curve). The second and higher order derivatives are also similarly motivated by some physical considerations (like the acceleration) and some geometrical ideas (like the curvature of a curve), which we shall study in the remaining units of this block.

We shall introduce higher order derivatives in Sec.1 and 2. Leibniz Theorem which is given in Sec.4 gives us a formula for finding the higher derivatives of a product of two functions. In the later sections, we shall study some useful formulas, called series expansions. The significance of these expansions will become clearer in Unit 14.

Objectives

After reading this unit you should be able to:

- calculate higher order derivatives of a given function f
- use the Leibniz formula to find the nth derivatives of products of functions
- expand a function using Taylor's Maclaurin's series.

6.2 SECOND AND THIRD ORDER DERIVATIVES

Consider the function $f(x) = x^4$. You know that $f'(x) = 4x^3$. Now, this f' is again a polynomial function and hence, can be differentiated (see Example 5, Unit 3). We shall denote the derivative of f' by f''. Thus,

$f''(x) = 12x^2$. This $f''(x)$ is called the second derivative of the function f at the point x. It is also denoted by $\frac{d^2y}{dx^2}$ (read as d square y by d x square) or y_2 or $f^{(2)}$ or D^2y.

Let us differentiate f''. We get $f'''(x) = 24x$, where f''' denotes the derivative of f , or the third derivative of f. Other notations for $f'''(x)$ are $\frac{d^3y}{dx^3}$ or y_3 or $f^{(3)}$ or D^3y. Differentiating f''', we get the fourth derivative of f, $f^{(4)}(x) = \frac{d^4y}{dx^4} = y_4 = 24$.

Thus, repeatedly differentiating (if possible) a given function f, we get the second, third, fourth,..... derivatives of f. These are called the higher order derivatives of f.

If n is any positive integer, then the n^{th} derivative of f is denoted by $f^{(n)}$ or by $\frac{d^ny}{dx^n}$ (read as d n y by d x n) or by y_n or D^ny.

Note that in the notation $f^{(n)}$ the bracket is necessary to distinguish it from f^n, that is, f raised to the power n. This process of differentiating again and again, in succession, is called **successive differentiation**.

We have already seen that there are functions f that are not differentiable. In other words f' need not always exist. Similarly even when f' exists it is possible that f'' does not exist (see Example 3 near the end of this section). In general, for each positive integer n there are functions f such that $f^{(n)}$ exists, but $f^{(n+1)}$ does not exist. However, many functions that we consider in these sections possess all higher derivatives.

A twice differentiable function is a function f such that f'' exists. Let n be a positive integer. A function f such that $f^{(n)}$ exists is called an n-times differentiable function. If $f^{(n)}$ exists for every positive integer n, then f is said to be an infinitely differentiable function.

Now we give some simple examples of higher derivatives.

Example 1 If we are given that the third derivative of the function

$f(x) = ax^3 + bx + c$ has the value 6 at the point $x = 1$, can we find the value of a?

Here, $f(x) = ax^3 + bx + c$

Differentiating this we get

$f'(x) = 3ax^2 + b$

Differentiating this again, we get

$f''(x) = 6ax$

Differentiating once again, we get

$f^{(3)}(x) = 6a$

Taking the value at $x = 1$,

$f^{(3)}(1) = 6a$

It is given that $f^{(3)}(1) = 6$.

Thus $6a = 6$. Therefore $a = 1$.

Example 2 If $y = 2\sin x + 3\cos x + 5$, let us prove that $y_2 + y = 5$.

Such equations involving the derivatives are known as differential equations.

Now, $y = 2\sin x + 3\cos x + 5$

$y_1 = 2\cos x - 3\sin x \qquad y_2 = -2\sin x - 3\cos x$

$\therefore \quad y_2 + y = -2\sin x - 3\cos x + 2\sin x + 3\cos x + 5 = 5$

The example below gives a function f for which f' exists but f'' does not exist.

Example 3 Consider the function $f(x) = x|x|$ for all x in R.

Recall (Unit 1) that $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

The function f(x) can be rewritten as

$$f(x) = \begin{cases} x^2 \text{ if } x \geq 0 \\ -x^2 \text{ if } x < 0 \end{cases}$$

At points other than 0 we have

$f'(x) = 2x$ if $x > 0$

$f'(x) = -2x$ if $x < 0$

At $x = 0$, the right derivative of f,

$$Rf'(0) = \lim_{h \to 0^+} \frac{h^2 - 0^2}{h} = \lim_{h \to 0^+} h = 0, \text{ and}$$

the left derivative of f,

$$Lf'(0) = \lim_{h \to 0^-} \frac{-h^2 - 0^2}{h} = \lim_{h \to 0^-} -h = 0.$$

Therefore $f'(0) = 0$.

Thus, $f'(x) = 2|x|$ for all x in R.

We have already seen in Example 7, Unit 3 that the absolute value function |x| fails to be differentiable at 0. Therefore, f' is not differentiable at $x = 0$. That is, $f^{(2)}(0)$ does not exist.

Try some exercises before going any further.

E 1) Find the second derivatives of the following functions.

a) $f(x) = x^4 - 4$

b) $y = e^{2x}$

E 2) Find $f^{(3)}(\pi/4)$ for the following functions.

a) $f(x) = \sec x$

b) $f(x) = \sin 2x + \cos 2x$

E 3) Prove that the following functions satisfy the differential equations shown against them.

a) $y = \sin x$; $y_4 = y$

(b) $y = \cos x$; $(y_2)^2 + (y_3)^2 = 1$

E 4) Find the value of integer k in each of the following

a) $f(x) = \sin kx$ and $f^{(2)}(\pi/6) = 2\sqrt{3}$

b) $f(x) = x^k + kx^2 + 1$ and $f^{(2)}(1) = 12$

6.3 nth ORDER DERIVATIVES

Let n be a natural number. We have already defined the n^{th} derivative of a function in Sec.2.

When a function f is given by a formula, it is often necessary to express its n^{th} derivative also by a formula using f and n. Usually, one can guess $f^{(n)}$ after working out $f^{(1)}$, $f^{(2)}$ and $f^{(3)}$. However, a rigorous proof would require an application of the principle of mathematical induction.

In the examples below we shall derive formulas for the n^{th} derivative of various functions. Study them carefully as we shall be using them in later sections.

But, first let us recall the principle of mathematical induction.

If $\{P_n\}$ is a sequence of propositions (statements) satisfying

i) P_N is true (usually $N = 1$),

ii) The truth of P_i implies the truth of P_{i+1}, $i \geq N$,

then P_n is true for all $n \geq N$.

We shall apply this principle in the examples that follow.

Example 4 We shall prove here that the n^{th} derivative of the polynomial function x^m is

$$\frac{d^n x^m}{dy^n} = \begin{cases} m(m-1) \ldots (m-n+1)\, x^{m-n}, \text{ if } n \leq m, \text{ and} \\ 0, \text{ if } n > m \end{cases}$$

Let us denote by P_n the statement

$$\frac{d^n x^m}{dy^n} = \begin{cases} m(m-1) \ldots (m-n+1)\, x^{m-n} \text{ if } n \leq m \\ 0, \text{ if } n > m. \end{cases}$$

Note that the product $m(m-1) \ldots (m-n+1)$ has n factors. When $n = 1$, only one factor, namely, m is taken.

Thus $P_1 \left(\frac{dx^m}{dy} = mx^{m-1} \right)$ is true.

Suppose we have proved for some n that P_n is true. This means that the n^{th} derivative of x^m is

$$= \begin{cases} m(m-1) \ldots m(m-n+1)\, x^{m-n} \text{ if } n \leq m, \text{ and is} \\ 0 \text{ if } n > m. \end{cases}$$

When $n = m$, the n^{th} derivative is constant, because $x^{m-n} = x^0 = 1$; therefore the $(n+1)^{th}$ derivative is zero.

Check that the conditions "$n = m$ or $n > m$" and "$n + 1 \geq m$" are equivalent.

Then the $n + 1^{th}$ derivative of x^m is

= the derivative of the n^{th} derivative of x^m

$$= \begin{cases} m(m-1)\ldots(m-n+1)(m-n)x^{m-n-1} \text{ if } n < m \\ 0 \text{ if } n = m \\ 0 \text{ if } n > m. \end{cases}$$

$$= \begin{cases} m(m-1)\ldots(m-n+1)(m-n)x^{m-(n+1)} \\ \qquad\qquad \text{if } n+1 \leq m \\ 0 \text{ if } n+1 > m. \end{cases}$$

This means that the truth of P_n implies the truth of P_{n+1}. Therefore, by the principle of mathematical induction, P_n is true for all $n \geq 1$. Hence our result is true for all natural numbers n.

Remark 1 When $n = m$, the n^{th} derivative of x^m is

$= m(m-1)\ldots(m-n+1)x^{m-n} = m(m-1)\ldots 3.2.1$. This is the same as m!.

Example 5 If $f(x) = \ln(1 + x)$, let us find $f^{(n)}(x)$.

Differentiating $f(x) = \ln(1 + x)$, we get $f'(x) = \frac{1}{1+x}$.

Differentiating again, $f^{(2)}(x) = -\frac{1}{(1+x)^2}$

Differentiating once again, $f^{(3)}(x) = \frac{2}{(1+x)^3}$

Can you guess $f^{(n)}(x)$ now? If you have guessed correctly, you must have arrived at these conclusions.

i) The denominator of $f^{(n)}(x)$ is $(1 + x)^n$.

ii) Its sign is positive or negative according as n is odd or even.

iii) Its numerator has $(n - 1)!$ Do not think that it is merely $(n - 1)$. There is a factorial symbol too. To be convinced of this, calculate $f^{(4)}(x)$ and see.

Therefore our guess is $f^{(n)}(x) = \frac{(-1)^{n-1} \times (n-1)!}{(1+x)^n}$ ..... (1)

This guess remains to be proved. A proof is necessary because there could exist many other formulas for $f^{(n)}(x)$ that coincide with the correct answer when $n = 1, 2$ or 3. For example, if we omit the factorial symbol, we get one such formula. But we have already mentioned that this formula does not hold for $f^{(4)}(x)$. So, let's try to prove (1).

We first note that it is true for $n = 1, 2$ and 3 as we have seen in the beginning.

Assume that it is true for $n = m$, that is,

$$f^{(m)}(x) = \frac{(-1)^{m-1}(m-1)!}{(1+x)^m}$$

Differentiating this we get,

$$f^{(m+1)}(x) = (-1)^{m-1}.(m-1)!\ \frac{-m}{(1+x)^{m+1}} = \frac{(-1)^{m-1}(-1) \times m(m-1)!}{(1+x)^{m+1}}$$

$$= \frac{(-1)^m (m)!}{(1+x)^{m+1}} = \frac{(-1)^{m+1-1}(m+1-1)!}{(1+x)^{m+1}}$$

This proves the guess for $n = m + 1$. Thus, assuming the truth of the formula (1) for $n = m$, we arrive at the truth of this formula for $n = m + 1$. Therefore by the principle of mathematical induction, our guessed answer is correct for all positive integers n.

Thus, when $f(x) = \ln(1 + x)$,

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

E E 5) If $y = (1 + x)^r$, where r is a real number, find y_n where n is a natural number $(n < r)$.

Example 6 If $f(x) = \cos 2x$, let us use the principle of mathematical induction to find a formula for $f^{(n)}(0)$.

We first find $f^{(n)}(x)$ when $n = 1, 2, 3, 4$.

We have $f(x) = \cos 2x$.

Differentiating this successively, we get

$f^{(1)}(x) = -2 \sin 2x$

$f^{(2)}(x) = -4 \cos 2x$

$f^{(3)}(x) = 8 \sin 2x$

$f^{(4)}(x) = 16 \cos 2x$

We see that in the formula for $f^{(n)}(x)$, we have to have

i) a sign (positive or negative),

ii) a coefficient (some power of 2), and

iii) a trigonometric function ($\sin 2x$ or $\cos 2x$)

We observe that the first two terms carry negative sign, the next two carry positive sign, the next two negative and so on.

We also observe that sin and cos occur alternately. Therefore our guess is

$$f^{(n)}(x) = \begin{cases} -2^n \sin 2x & \text{if n is of the form } 4k+1 \\ -2^n \cos 2x & \text{if n is of the form } 4k+2 \\ 2^n \sin 2x & \text{if n is of the form } 4k+3 \\ 2^n \cos 2x & \text{if n is of the form } 4k \end{cases} \quad \ldots, (2)$$

Recall that
$\cos(\theta + \pi/2) = -\sin\theta$
$\cos(\theta + \pi) = -\cos\theta$
$\cos(\theta + 3\pi/2) = \sin\theta$

We can also write this in a compact form as

$f^{(n)}(x) = 2^n \cos(2x + n\pi/2)$ ..... (3)

You can easily check that (2) and (3) are equivalent by putting $n = 4k + 1, 4k + 2, 4k + 3$ and $4k$ in (3). We shall now prove formula (3).

We have already seen that it is true for $n = 1, 2, 3$ and 4. Suppose it is true for $n = m$, that is,

$f^{(m)}(x) = 2^m \cos(2x + m\pi/2)$.

Differentiating this we get,

$f^{(m+1)}(x) = -2^{m+1} \sin(2x + m\pi/2)$

or, $f^{(m+1)}(x) = 2^{m+1} \cos[2x + (m+1)\pi/2]$

$\cos(2x + (m+1)\pi/2)$
$= \cos(2x + m\pi/2 + \pi/2)$
$= -\sin(2x + m\pi/2)$

So here again we see that the truth of the formula (3) for $n = m$ implies its truth for $n = m + 1$.

Therefore by the principle of mathematical induction, we see that the guessed formula for $f^{(n)}(x)$ is true for all natural numbers n.

Now substitute $x = 0$. We obtain

$f^{(n)}(0) = 2^n \cos n\pi/2$

This is the required answer.

We can also use this method to prove a general result about the n^{th} derivative of a sum of two functions.

Example 7 If f and g are two functions from R to R and if both (differentiable, we can prove that

$(f + g)^{(n)} = f^{(n)} + g^{(n)}$.

We shall prove this result by induction.

When $n = 1$, this means $(f + g)' = f' + g'$

This has already been proved in Unit 3.

Suppose $(f + g)^{(m)} = f^{(m)} + g^{(m)}$ is true.

Differentiating this we get

$[(f + g)^{(m)}]' = [f^{(m)} + g^{(m)}]' = [f^{(m)}]' + [g^{(m)}]'$

This is the same as

$(f + g)^{(m+1)} = f^{(m+1)} + g^{(m+1)}$

Thus the result is true for $n = m + 1$. Therefore by the principle of mathematical induction

$(f + g)^{(n)} = f^{(n)} + g^{(n)}$ holds for all natural numbers n.

Remark 3 Similarly one can prove that

$(cf)^{(n)} = c. f^{(n)}$ holds for all natural numbers n and all scalars c. This fac[illegible] ed with Example 7 can be restated in the "linear algebra terminology" as :

The collection of n-times differentiate functions is a vector space under usual operatic[illegible].

Try to solve these exercises now.

E 6) Find the n^{th} derivative of the following functions :

a) $f(x) = (ax + b)^3$

b) $f(x) = (ax + b)^m$

c) $f(x) = e^x$

d) $f(x) = e^{kx}$

E 7) If $f(x) = \sin x$, prove that $f^{(n)}(x) = \sin[x + n\pi/2]$ holds for every natural number n.

E 8) If $y = \sin(ax + b)$, prove that for every positive integer n, we have
$y_n = a^n \sin[(n\pi/2) + ax + b]$

E E 9) Prove that the n^{th} derivative of the polynomial function
$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is the constant function $n!\,a_n$.

E E 10) If $y = \cos x$ and if n is any positive integer, prove that
$[y_n]^2 + [y_{n+1}]^2 = 1$.

6.4 LEIBNIZ THEOREM

In Unit 3 we have proved some rules regarding the derivatives of the sum, scalar multiple, product and quotient of two differentiable functions. These were

$(f + g)' = f' + g'$

$(cf)' = cf'$

$(fg)' = fg' + gf'$

$(f/g)' = \frac{gf' - fg'}{g^2}$ ($g(x) \neq 0$ anywhere in the domain)

In the last section we have seen (Example 7 and Remark 3) that the first two rules can be extended to the n^{th} derivatives if f and g are n-times differentiable functions. In this section we are going to extend the product rule of differentiation. We shall give a formula for the n^{th} derivative of the product of two functions.

The product rule for two functions u and v can also be written as

$(uv)_1 = u_1v + uv_1$

Now we look for a similar formula for $(uv)_2$, $(uv)_3$, etc.

u_n denotes the n^{th} order derivative of u.

But first let us recall the meaning of the notation C(n,r), where n and $r \in Z^+$ and $r \leq n$. This C(n,r) stands for the number of ways of choosing r objects from n objects. Sometimes it is also denoted by ${}^nC_r$ or $\binom{n}{r}$.

Also recall the formulas

i) $C(n,r) = \frac{n!}{r!\,(n-r)!}$

ii) $C(n,0) = C(n,n) = 1$

iii) $C(n,r) = C(n,n-r)$

iv) $C(n,r) + C(n,r+1) = C(n+1, r+1)$

These are combinatorial identities, true for all positive integers r and n with $r \leq n$.

Theorem 1 (Leibniz Theorem) Let n be a positive integer. If u and v are n times differentiable functions, then

$(uv)_n = C(n,0)\, u_n v + C(n,1)\, u_{n-1} v_1 + C(n,2)\, u_{n-2} v_2 + \ldots + C(n,n)\, u v_n$

Leibniz had stated this result in his first article on differential calculus which was published in 1684.

The pattern in the formula for $(uv)_n$ can be compared with the expansion of $(x+y)^n$. The coefficients are binomial coefficients and they appear in the same order as those in the expansion of $(x+y)^n$. The order of the derivative of u goes on decreasing one at a time, and the order of the derivative of v goes on increasing one at a time. The number of terms is $n + 1$.

Remark 4 We omit the proof of this theorem and merely indicate how this can be proved by induction on n. Firstly, when $n = 1$, the above formula is the same as the already known product formula, and therefore is true. Assuming that it is true for $n = m$, we can prove it for $n = m + 1$, by applying the product rule for each term of the expansion of $(uv)_m$ and by using the combinatorial identities mentioned. (See E 17) for more details.

We start with a simple and direct application of the formula.

Example 8 If $f(x) = x \sin x$, let us find the fourth derivative of f, using Leibniz Theorem.

We first observe that for $n = 4$, the Leibniz Theorem states

$$(u.v)_4 = C(4,0)\, u_4 v + C(4,1)\, u_3 v_1 + C(4,2)\, u_2 v_2 + C(4,3)\, u_1 v_3 + C(4,4)\, u v_4$$
$$= u_4 v + 4\, u_3 v_1 + 6\, u_2 v_2 + 4\, u_1 v_3 + u v_4.$$

In this problem we take $u = x$ and $v = \sin x$, so that $f = uv$

We have $u = x$, $u_1 = 1$, $u_2 = 0 = u_3 = u_4$

$v = \sin x$, $v_1 = \cos x$, $v_2 = -\sin x$, $v_3 = -\cos x$, $v_4 = \sin x$

Substituting these in the above formula, we get

$$f^{(4)} = (uv)_4 = 0 + 0 + 0 + 4(-\cos x) + 1.x.\sin x$$
$$= x \sin x - 4 \cos x$$

What happens if we attack the same problem directly without the use of Leibniz Theorem? We have

$f(x) = x \sin x$

Differentiating this, we get

$f'(x) = x \cos x + \sin x$ (by product rule)

Differentiating once again, we get

$$f''(x) = x(-\sin x) + 1 \cos x + \cos x$$
$$= 2 \cos x - x \sin x$$

Differentiating once again, we get

$$f^{(3)}(x) = -2 \sin x - (x \cos x + \sin x)$$
$$= -3 \sin x - x \cos x$$

Differentiating once again,

$$f^{(4)}(x) = -3 \cos x - [x(-\sin x) + \cos x]$$
$$= x \sin x - 4 \cos x$$

We notice that we obtain the same answer. In this direct method, we had to apply the product formula four times, once for each differentiation.

It is clear that when we want the n^{th} derivative for bigger values of n, Leibniz theorem provides an easier method to write down the answer, avoiding the difficulty of repeatedly applying the product formula.

Example 9 If $y = (\sin^{-1}x)^2$, prove that

$(1 - x^2)\, y_{n+2} - (2n + 1)\, x y_{n+1} - n^2 y_n = 0$ for each p[illegible]

Differentiating both sides of $y = (\sin^{-1}x)^2$, we get

$$y_1 = \frac{2 \sin^{-1}x}{\sqrt{1 - x^2}}$$

Squaring and crossmultiplying we get

$(1 - x^2)\, y_1^2 = 4 (\sin^{-1}x)^2 = 4y$

Differentiating once again, we get

$2 (1 - x^2)\, y_1 y_2 - 2x y_1^2 - 4\, y_1 = 0$

Dividing throughout by $2\, y_1$ gives us

$(1 - x^2)\, y_2 - x y_1 - 2 = 0$

Differentiating n times, using Leibniz Theorem for each of the first two terms we get

$(1 - x^2)\, y_{n+2} - C(n,1)\, 2x y_{n+1} - C(n,2)\, 2\, y_n - \{x y_{n+1} + C(n,1)\, y_n\} = 0$

That is,

$(1 - x^2)\, y_{n+2} - (2n + 1)\, x y_{n+1} - n^2 y_n = 0$

The following exercises will give you some practice in applying Leibniz Theorem.

E 11) State Leibniz Theorem when $n = 5$. That is,

$(u.v)_5 = ?$

E 12) Prove that when $n = 1$, Leibniz Theorem reduces to the product rule of differentiation.

E 13) Find the third derivative of $\sin^4 x$ using Leibniz Theorem. Find the same directly also and verify that you obtain the same answer.

E E 14) If $f(x) = x e^x$, find the sixth derivative of f, using Leibniz formula.

E E 15) Find the n^{th} derivative $x^3 \ln x$

E E 16) If $y = e^{ax}x^2$ prove that

$$y^{(n)} = e^{ax}[a^n x^2 + 2na^{n-1}x + n(n-1)a^{n-2}]$$

E E 17) a) Write down Leibniz formula for $(u.v)_m$

b) Differentiate it term by term and obtain
$(uv)_{m+1} = C(m,0)\, u_{m+1}v + C(m,1)\,(u_m v_1 + u_{m-1}v_2) + ... + C(m,m)\, uv_{m+1}$.

c) Deduce that
$(uv)_{m+1} = C(m,0)\, u_{m+1}v + [C(m,0) + C(m,1)]\, u_m v_1 + [C(m,1) + C(m,2)]\, u_{m-1}v_2 + ... + [C(m,m-1) + C(m,m)]\, u_1 v_m + C(m,m)\, uv_{m+1}$.

d) Deduce from (c) the Leibniz formula for $(uv)_{m+1}$.

E E 18) Using Leibniz Theorem and induction, prove that
$(x^n)^{(n)} = n!$ for all natural numbers n.

6.5 TAYLOR'S SERIES AND MACLAURIN'S SERIES

In this section we obtain series expansions for many important functions. For this, we use higher derivatives.

You must have come across the following series :

i) Exponential Series :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$

ii) Logarithmic Series :

$$\ln\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^{n+1}\frac{x^n}{n} + ..., |x| < 1$$

iii) Geometric Series :

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots \text{ provided } |x| < 1$$

iv) Binomial Series :

$$(1 + x)^r = 1 + rx + \frac{r(r-1)}{1.2} x^2 + \ldots \text{ provided } |x| < 1$$

We observe that each of them is of the form

$f(x) = a_0 + a_1 x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$, where $a_0, a_1, a_2, \ldots, a_n, \ldots$ are some constants.

We ask ourselves the questions : Is there anything else common to these four examples? Is it possible to express $a_0, a_1, \ldots, a_n, \ldots$ in terms of the function f? Our answer is : Yes. In all these examples,

$$a_0 = f(0)$$

$$a_1 = \frac{f'(0)}{1!}$$

$$a_2 = \frac{f^{(2)}(0)}{2!}$$

$$\vdots$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$\vdots$$

In other words, the series is of the form

$$\boxed{f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n + \ldots}$$

Brook Taylor (1685-1731) and Colin Maclaurin (1698-1746) were both disciples of Newton. Taylor first published his series in a paper in 1715. Maclaurin used Taylor's series as a fundamental tool in his work on calculus.

We shall prove this for the above four instances, in the examples worked out below. This expansion is called **Taylor's series for f around zero**. It is also known as **Maclaurin's series for f**. The name "Taylor's Series for f around zero" suggests that there may be a Taylor's series for f around $x_0 (x_0 \neq 0)$. But in this course we shall restrict ourselves only to the series around zero. This series expansion makes sense only when f is infinitely differentiable at zero. It is valid for many important functions (though not for all functions). You will learn more about the validity of these series in the course on real analysis. In this section, you should train yourselves to write down Maclaurin's series for many functions.

We have said above that the function f should be infinitely differentiable, that is, it should have derivatives of all orders. How do we check this condition? For some functions it is not difficult. For example, we have

$$\frac{d^n}{dx^n} (\sin x) = \sin\left(x + \frac{n\pi}{2}\right), \quad \frac{d^n}{dx^n} (\cos x) = \cos\left(x + \frac{n\pi}{2}\right),$$

$$\frac{d^n}{dx^n} (e^{ax}) = a^n e^{ax},$$

$$\frac{d^n}{dx^n} (\cosh x) = \frac{d^n}{dx^n} \left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2} [e^x + (-1)^n e^{-x}]$$

$$= \begin{cases} \cos hx, \text{ if n is even} \\ \sin hx, \text{ if n is odd} \end{cases}$$

[illegible] we can say that the derivat[illegible] orders exist for all values of x and we can [illegible] Taylor's series.

Example 10 Let us verify that the known series expansion of e^x is the same as its Maclaurin's series.

Maclaurin's series for e^x is

$$f(0) + \frac{f'(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

Where $f(x) = e^x$

Now, $f(0) = e^0 = 1$

Also, $f'(x) = e^x \therefore f'(0) = 1$.

In fact, we know that $f^{(n)}(x) = e^x$ for all natural numbers n, which means $f^{(n)}(0) = e^0 = 1$ for all natural numbers n. Substituting these values of $f(0), f'(0), \ldots, f^{(n)}(0)$ in Maclaurin's series we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$$

which is the known expansion for e^x.

Example 11 Obtain Taylor's series for $\ln(1 + x)$ around zero. Let $f(x) = \ln(1 + x)$

Then we have already seen in Example 5, that

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$$

Therefore, $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$$\therefore \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n}$$

Therefore Taylor's series around zero is

$$\ln(1 + 0) + \frac{(-1)^{1-1}}{1} x + \frac{(-1)^{2-1}}{2} x^2 + \cdots + \frac{(-1)^{n-1}}{n} x^n + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}}{n} x^n + \cdots$$

We note that this is the same as the already known logarithmic series.

Example 12 Now let us write down Maclaurin's series (or Taylor's series around zero) for $1/(1 - x)$.

Let $f(x) = \frac{1}{1-x}$. Then $f(0) = \frac{1}{1-0} = 1$

$f'(x) = \frac{1}{(1-x)^2}$; $f'(0) = 1$

$f^{(2)}x = \frac{2}{(1-x)^3}$; $f^{(2)}(0) = 2$

We can prove by induction that

$f^{(n)}x = \frac{n!}{(1-x)^{n+1}}$ and therefore $f^{(n)}(0) = n!$

Therefore Maclaurin's series is

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

Note that this agrees with what we already know, namely that the sum of the geometric series $1 + x + x^2 + \ldots + x^n + \ldots$ is $1/(1 - x)$.

Example 13 Suppose we want to write down Taylor's series for $(1 + x)^r$ around zero, where r is a fixed real number.

Let $f(x) = (1 + x)^r$. Then $f(0) = 1$

$f'(x) = r(1 + x)^{r-1}$; $f'(0) = r$.

$f^{(2)}(x) = r(r-1)(1 + x)^{r-2}$; $f^{(2)}(0) = r(r-1)$

We can prove by induction that

$f^{(n)}(x) = r(r-1)\,.....\,(r-n+1)(1+x)^{r-n}$

$f^{(n)}(0) = r(r-1)\,.....\,(r-n+1)$

Therefore Taylor's series around zero is

$$1 + \frac{r}{1!}x + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\ldots(r-n+1)}{n!}x^n + \cdots$$

Note : This is the same as the binomial series that we already know. This expansion is valid only when $|x| < 1$. The reason for this will be clear when you study the course "Real Analysis". When $r = -1$, this binomial series becomes

$$(1+x)^{-1} = 1 + \frac{-1}{1!}x + \frac{(-1)(-2)}{2!}x^2 + \ldots + \frac{(-1)(-2)\ldots(-n)}{n!}x^n + \ldots$$

$$= 1 - x + x^2 - \ldots.. + (-1)^n x^n + \ldots..$$

Note that Example 12 follows from this on replacing x by $-x$ throughout.

So far we have seen that the four known series occur as Taylor's series. In the next two examples, we find that we can write down similar series even for functions like sin x and cos x.

Example 14 Let us write down Maclaurin's series for the function sin x.

Let $f(x) = \sin x$. Then we have already seen in E 7)

$$\text{that } f^{(n)}(0) = \begin{cases} 0 \text{ if n is even} \\ 1 \text{ if n is divisible by 4.} \\ -1 \text{ otherwise} \end{cases}$$

$$f^{(n)}{}_{(o)} = \sin\frac{n\pi}{2}$$

We see that, as n varies over 0, 1, 2, 3, 4, 5, 6, 7 ....,

$f^{(n)}(0)$ takes the values 0, 1, 0, -1, 0, 1, 0, -1, ...

Therefore Maclaurin's series for sin x is

$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \cdots$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Example 15 To find Taylor's series for cos 2x around zero, let us write $f(x) = \cos 2x$. We have already seen in Example 6 that

$$f^{(n)}(0) = 2^n\cos\frac{n\pi}{2}$$

Therefore, Taylor's series around zero is

$$\cos 2x = 1 + \frac{0.x}{1!} - \frac{2^2x^2}{2!} - \frac{0.x^3}{3!} + \frac{2^4x^4}{4!} - \frac{0.x^5}{5!} + \cdots$$

$$= 1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \cdots$$

Example 16 Suppose we want to

a) write down the first four terms of Maclaurin's series for tan x.
b) write down the first three non zero terms of this series.

Let $f(x) = \tan x$. Then $f(0) = 0$

$f'(x) = \sec^2 x$, $f'(0) = 1$

$f''(x) = 2\sec^2 x \tan x$, $f''(0) = 0$

$f^{(3)}(x) = 2\sec^4 x + 4\sec^2 x\tan^2 x$

$\quad = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$

$f^{(3)}(0) = 2(1+0) = 2$

Therefore the first four terms of Maclaurin's series for tan x are given by

$0, \frac{1}{1!}x, \frac{0}{2!}x^2, \frac{2}{3!}x^3,$

Hence Maclaurin's series for tan x is

$x + \frac{x^3}{3} + \cdots$

Now, we want the next non-zero term.

We have $f^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$ and

$f^{(4)}(0) = 0.$

Next $f^{(5)}(x) = 16\sec^6 x + 64\sec^4 x \tan^2 x + 24\sec^4 x \tan^2 x + 16\sec^2 x \tan^4 x$

This means $f^{(5)}(0) = 16.$

Thus we have

$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

Example 17 If in Maclaurin's series for sin kx, the coefficient of x^3 is given to be $-6k$, let us find all possible values of k.

Maclaurin series for sin kx is given by

$\sin kx = \frac{kx}{1!} - \frac{k^3x^3}{3!} + \cdots$, since $f'(x) = k\cos kx$

$f^{(2)}(x) = -k^2 \sin kx$

$f^{(3)}(x) = -k^3 \cos kx$, where $f(x) = \sin kx$

The coefficient of x^3 is $-(k^3/6)$

Therefore $-(k^3/6) = -6k$

This gives the equation $k(k^2 - 36) = 0$

The roots are $k = 0, 6$ or -6.

Thus, 0, 6 and -6 are all the possible values of k such that the coefficient of x^3 in Maclaurin's series for sin kx is $-6k$.

E 19) Write down Maclaurin's series for the following:

a) $\frac{1}{(1+x)^2}$

b) $(x-2)^2 + 1$

c) cos x

d) $1/(1-2x)$

E E 20) Write down the first three non-zero terms in Maclaurin's series of the following.

a). $\sin 3x$

b) $\ln(1-x)$

E E 21) Find the coefficient of x^9 in Maclaurin's series for the functions.

a) $\cos 2x$

b) $\sin\left(x + \frac{\pi}{4}\right)$

E 22) If Maclaurin's series for sin x is differentiated term by term, do you get Maclaurin's series for cos x?

E 23) If Maclaurin's series for e^x is differentiated term by term, we get the same series again. Prove this.

E 24) Consider the function $y = a + \tan^{-1} bx$ where a and b are fixed real numbers. We are given that its Taylor's series around zero is $2 + 3x - 9x^3 + \ldots$

Find the values of a and b.

E 25) Find the coefficient of x^3 in Taylor's series around zero for the function $\sin^{-1} x$.

Some General Remarks on Taylor's and Maclaurin's Series.

Though we have obtained infinite series for many functions, it is necessary to give a note of caution. These infinite series need not be valid for all values of x, and as such, these have to be used with care. In the course on real analysis, you will be able to study the conditions under which these series are valid.

6.6 SUMMARY

In this unit, we have

1) introduced higher order derivatives,
2) derived a formula (Leibniz's Theorem) for the n^{th} derivative of a product of two functions.
   $(u\,v)_n = C(n, 0)\, u_n\, v + C(n, 1)\, u_{n-1}\, v_1 + C(n, 2)\, u_{n-2}\, v_2 + \ldots + C(n, n)\, u\, v_n$.
3) written Taylor's series around zero/Maclaurin's series of a number of functions by using the formula

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

6.7 SOLUTIONS AND ANSWERS

E 1) a) $6x$ b) $4e^{2x}$

E 2) a) $11\sqrt{2}$ b) 8

E 3) a) $y = \sin x \implies y_4 = \sin x = y$

b) $y = \cos x \implies y_1 = -\sin x,\ y_2 = -\cos x,\ y_3 = \sin x$
$\implies (y_2)^2 + (y_3)^2 = \cos^2 x + \sin^2 x = 1.$

E 4) a) $f(x) = \sin kx \implies f^{(12)}(x) = -k^2 \sin kx$
$\implies f^{(12)}(\pi/6) = -k^2 \sin k\pi/6$

Now, $-k^2 \sin k\pi/6 = 2\sqrt{3} \implies \sin k\,\pi/6 = -2\sqrt{3}/k^2$

$k \neq 0$, since $k = 0 \implies 0 = 2\sqrt{3}$, which is impossible.

Since $-1 < \sin k\,\pi/6 < 0$, $-\pi < k\,\pi/6 < 0$
$\implies k = -1$ or -2

Out of these, $k = -2$ is the value which satisfies $\sin k\,\pi/6 = -2\sqrt{3}/k^2$.

b) $f(x) = x^k + kx^2 + 1 \implies f'(x) = kx^{k-1} + 2kx$
$f''(1) = k(k-1) + 2k = 12 \implies k = 3$ or -4

E 5) $y_n = r(r-1) \ldots (r-n+1)(1+x)^{r-n}$

E 6) a) $= \begin{cases} \frac{3!\,a^n}{(3-n)!}(ax+b)^{3-n};\ n \le 3 \\ 0, \text{ if } n > 3. \end{cases}$

b) $= \begin{cases} \frac{m!\,a^n}{(m-n)!}(ax+b)^{m-n} \text{ if } n \le m \\ 0, \text{ if } n > m. \end{cases}$

c) e^x d) $k^n e^{kx}$

E 7) $f(x) = \sin x \implies f'(x) = \cos x,\ f''(x) = -\sin x,$
$f^{(3)}(x) = -\cos x,\ f^{(4)}(x) = \sin x$ and so on. So Our guess is that $f^{(n)}(x) = \begin{cases} \cos x \text{ if } n = 4k+1 \\ -\sin x \text{ if } n = 4k+2 \\ -\cos x \text{ if } n = 4k+3 \\ \sin x \text{ if } n = 4k \end{cases}$

Now use the principle of mathematical induction to prove that $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$ as in Example 6.

E 10) $y = \cos x \implies y_1 = -\sin x,\ y_2 = -\cos x,\ y_3 = \sin x,\ y_4 = \cos x$ and so on.
$y_n = \cos(x + n\pi/2)$
$\implies y_{n+1} = -\sin(x + n\pi/2)$
$\implies y_n^2 + y_{n+1}^2 = \cos^2(x + n\pi/2) + \sin^2(x + n\pi/2) = 1.$

E 11) $(uv)_5 = u_5 v + 5u_4 v_1 + 10u_3 v_2 + 10u_2 v_3 + 5u_1 v_4 + v_5$

E 12) $(uv)_1 = u_1 v + u v_1$ which is the product rule of differentiation.

E 13) $\frac{d^3(\sin^2 x)}{dx^3} = \frac{d^3}{dx^3}(\sin x.\sin x) = -\cos x \sin x - 3\sin x \cos x$
$- 3\sin x \cos x - \cos x \sin x = -8 \sin x \cos x$

$\frac{d}{dx}(\sin^2 x) = 2\sin x \cos x$

$\implies \frac{d^2}{dx^2}(\sin^2 x) = 2(\cos^2 x - \sin^2 x)$

$\implies \frac{d^3}{dx^3}(\sin^2 x) = -8 \sin x \cos x$

E 14) $e^x(x+6)$

E 15) $\frac{(-1)^{n+1}x^3}{x^n}[(n-1)! - 3C(n,1)(n-2)! + 6C(n,2)(n-3)!$
$- 6C(n,3)(n-4)!]$

E 19) a) $1 - 2x + 3x^2 - 4x^3 + \cdots$

b) $5 - 4x + x^2 + 0.x^3 + 0.x^4 + \cdots$

c) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

d) $1 + 2x + 2^2x^2 + 2^3x^3 + \cdots$

E 20) a) $3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!}$

b) $-x - \frac{x^2}{2} - \frac{x^3}{3}$

E 21) a) 0 b) $\frac{1}{9!\sqrt{2}}$

E 22) Yes

E 24) We take that $\tan^{-1} bx$ always takes values between $-\pi/2$ and $\pi/2$. Then, $a = 2$, $b = 3$

E 25) 1/6

UNIT 7 THE UPS AND DOWNS

Structure

7.1 INTRODUCTION

Why do drops of oil on the surface of water coalesce? Why do honeycombs have hexagonal cells? Why is a drop of water spherical? Why does a red corpuscle in blood have the shape of a biconcave disc? The answers to these questions are closely related to minimum and maximum values of some functions. Drops of oils tend to coalesce so as to minimise total surface tension. The scheme of hexagonal cells enables bees to store a fixed amount of honey by using the minimum amount of wax for sealing. A drop of water is spherical because a sphere is the shape which encloses a given volume with minimum surface area. The oxygen carrier red corpuscle, on the other hand, is in the shape of a biconcave disc so as to maximise the surface area. It enables our system to carry the maximum amount of oxygen on the surface of a fixed amount of blood.

In this unit we shall discuss an important technique involved in solving the problem of maximising or minimising various functions. This technique, as you will soon see, involves the use of derivatives which you studied in Units 3-6. We shall also discuss Rolle's theorem and the mean value theorem which have very important applications as you will see further in your study of calculus.

Objectives

After having gone through this unit, you should be able to

- obtain the maximum and minimum values of some functions
- solve practical problems of maxima-minima
- state Rolle's theorem and the mean value theorem
- find the points of inflection and the curvature of a curve,
- determine whether a given function is concave or convex or neither in a given interval.

7.2 MAXIMA-MINIMA OF FUNCTIONS

Look at the points P and Q in the graphs in Fig. 1. How are they different from other points on the graphs?

We could describe Q's as the peaks or hill-tops and the P's as the valley-bottoms. Using the language of mathematics, we could say that each P has the property that the value of the function f at P is smaller than the value of f at neighbouring points. Similarly, the peaks Q are distinguished by f having a maximum value there when compared to the values at near-by points.

But before proceeding any further, let us define maximum and minimum points of a function precisely.

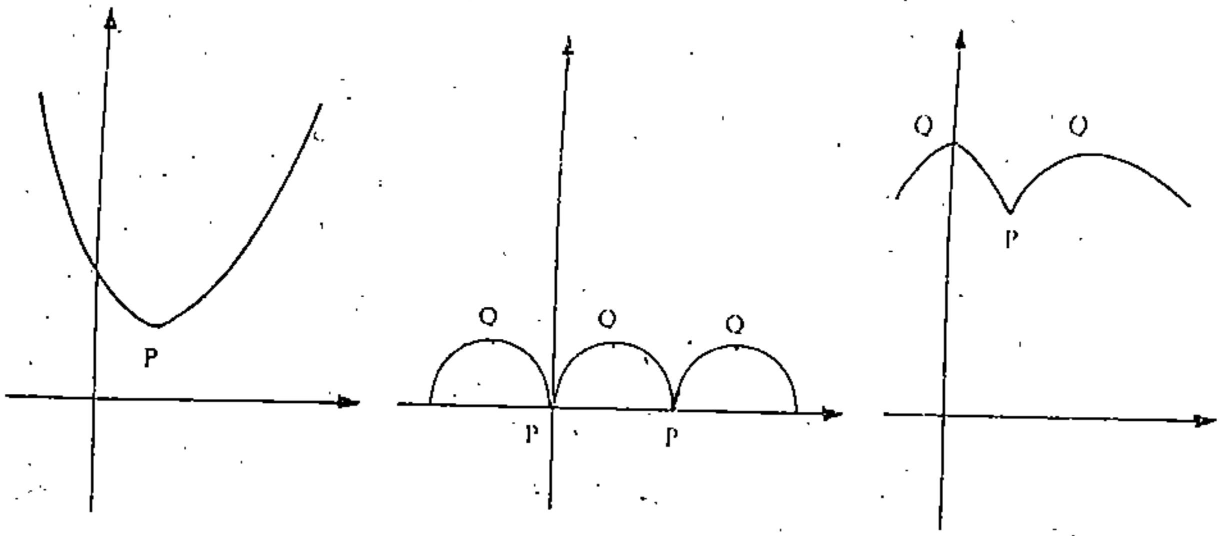


Fig. 1

7.2.1. Definitions and Examples

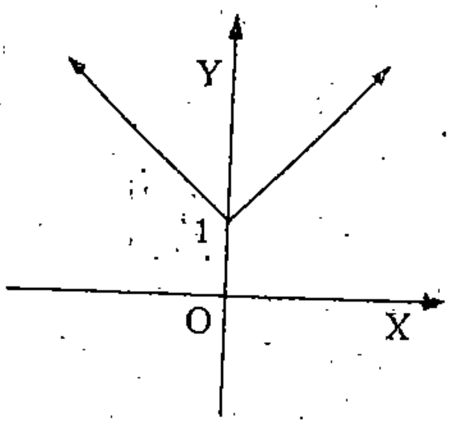
Definition 1 : A point c is said to be a maximum point for a function f if there exists a $\delta > 0$, such that
$f(c) \geq f(x)$ for all x such that $|x - c| < \delta$.
f is said to have a maximum at c, and f(c) is called a maximum value of f.

$]x - \delta, x + \delta[$, that is, the set of points whose distance from x is less than a positive number δ is called a neighbourhood of x.

We can similarly define a minimum value of f.
f is said to have a minimum at a point c of its domain provided there exists a $\delta > 0$ such that
$f(c) \leq f(x)$ whenever $|x - c| < \delta$.

A maximum or a minimum value is known as an extreme value.

Example 1 The function $f(x) = |x| + 1$ has a minimum value at $x = 0$. You can see from Fig. 2 that $f(0) = 1$ and $f(x) > 1$ for all x in any every neighbourhood $]0 - \delta, 0 + \delta[$ of 0. The minimum value of f is thus $f(0) = 1$. Do you agree that this function has no maximum value?



ig. 2 : Graph of $f(x) = |x| + 1$

i this example we see that $f(x) \geq f(0)$ in every neighbourhood of 0. But to prove that f(0) a minimum value, we need to find only one such neighbourhood.

ur next example shows that Definition 1 cannot be applied to functions defined on osed intervals if their extrema occur at either of the end points of the interval.

xample 2 Consider the function $f(x) = x + 2, \forall x \in [0, 2]$.
rom its graph (Fig. 3), it is clear that x = 0 is a minimum point and . = . is a maximum oint of this function. In fact, you will see that
$= f(0) \leq f(x) \leq f(2) = 4 \forall x \in [0, 2]$

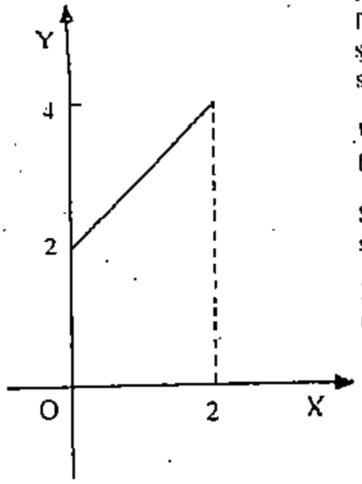


Fig. 3

Note that here we cannot find a $\delta > 0$ such that $f(0) \le f(x) \forall x \in]-\delta, \delta[$ for the simple reason that f(x) is not defined for $x < 0$. The same argument holds good for the maximum. Since f(x) is not defined for $x > 2$, we cannot find any $\delta > 0$, such that $f(x) \le f(2) \forall x \in]2-\delta, 2+\delta[$. This means f does not have extrema at 0 or 2 in the sense of Definition 1. How do we resolve this paradox? We modify Definition 1 to suit such cases.

Let f be defined on [a, b]. We shall say that f has a maximum (minimum) at a if we can find $\delta > 0$, such that $f(a) \ge f(x)$ ($f(a) \le f(x)$) for all $x \in [a, a+\delta[$.

Similarly, we shall say that f has a maximum (minimum) at b, if we can find $\delta > 0$ such that $f(b) \ge f(x)$ ($f(b) \le f(x)$) for all $x \in]b-\delta, b]$.

Example 3 The function $f(x) = \sin x$ has a maximum value at several points. Fig. 4 shows the graph of this function. Can you see that these points are $\pi/2, 5\pi/2, \ldots, -3\pi/2, -7\pi/2, \ldots$?

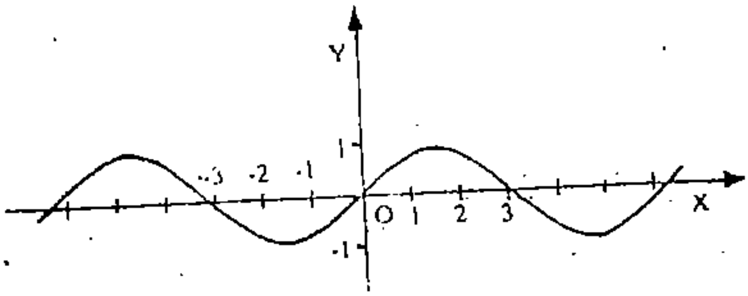


Fig. 4: Graph of f(x) = sin x

In general, the sine function has a maximum at each of the points $2n\pi + (\pi/2)$, n being any integer. The maximum value is $\sin[2n\pi + (\pi/2)] = 1$. This function has a minimum value too at several points. What are these points? Do you agree that the minimum value at these points is -1? We can now say that this function has an extremum at all points $n\pi + \pi/2$, n being any integer.

The functions in the examples considered so far were all continuous. For such functions, a valley in the graph of the function indicates a minimum and a peak points to a maximum. But we can talk about maximum and minimum values of non-continuous functions too. Here, we may find extreme values which are neither at a peak nor at a valley-bottom, as in Example 4.

Example 4 Consider the function f defined as follows :

$$f(x) \quad \begin{cases} -(x^2+1) \text{ if } x \le 1, \\ (x-3)^2 \text{ if } x > 1 \end{cases}$$

This function has three extreme points, P, Q and R (see Fig. 5). You should be able to see for yourself that P is a maximum and R is a minimum point of f.

The function also has a minimum at Q, that is, at $x = 1$. Let us see why. Consider the neighbourhood $\left]1-\frac{1}{2}, 1+\frac{1}{2}\right[$, that is, $\left]\frac{1}{2}, \frac{3}{2}\right[$ of 1. The function in this interval is defined as follows :

$$f(x) = \begin{cases} -1-x^2, \text{ when } \frac{1}{2} < x \le 1 \\ (x-3)^2, \text{ when } 1 < x < 3/2 \end{cases}$$

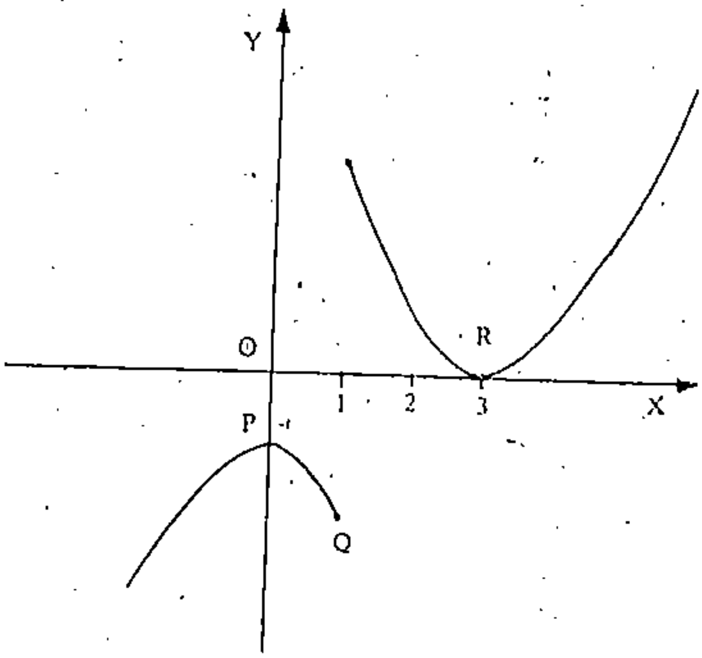


Fig. 5

Thus $f(1) = -1 - 1^2 = -2$.

For $\frac{1}{2} < x < 1$, $x^2 < 1 \implies -x^2 > -1$

$$\implies -1 - x^2 > -2$$

$$\implies f(x) > f(1)$$

Now, when $1 < x < 3/2$, $f(x)$ is positive, and therefore is greater than $f(1)$. Hence $f(x) \geq f(1)$ whenever $x \in]1/2, 3/2[$.

Thus $f(1) = -2$ is a minimum value of the function.

Remark 1 We found in the above example that $f(0) = -1$ was a maximum value of f whereas $f(3) = 0$ was a minimum value. Did you find it hard to swallow that a maximum value of the function is smaller than a minimum value? If yes, recall our definition of an extreme value. We were concerned with the values of the function only in a neighbourhood (that is, points near-by) of the extreme point. Thus, the concept of maxima-minima is essentially a local phenomenon. What happens globally or elsewhere was not under consideration. For this reason, some people use the terms local (or relative) maximum and local (or relative) minimum instead of maximum and minimum.

A value $f(c)$ such that $f(c) \geq f(x)$ for all x in the domain of the function, is then called an absolute (or global) maximum. Similarly, if $f(b) \leq f(x)$ for all x in the domain, then $f(b)$ is called the absolute (or global) minimum. Therefore, a function may have many local minima or maxima, but it can have only one absolute minimum or maximum. In the light of this we see that $f(0) = 1$ is the absolute minimum value for the function in Example 1.

E 1) Find the maxima, minima of each of the following functions. If a function has no maximum/minimum say so.

i) $f(x) = 5$, for all $x \in R$.

ii) $f(x) = x$, for all $x \in R$.

iii) $f(x) = x$, for $0 < x < 1$.
[Be careful! 0 and 1 are NOT in the domain of the function!]

iv) $f(x) = x^2$, for all $x \in R$.

v) $f(x) = \sqrt{x}$, for all $x \in [4, 16]$

7.2.2 A Necessary Condition for the Existence of Extreme Points

So far, we have used the graph of a function as an aid to discovering extreme points. Clearly, this is not going to be feasible in all circumstances. Drawing graphs may be time-consuming and cumbersome. Certainly there must exist quicker and neater techniques. In this subsection we describe an analytic method of finding extrema. But first, a definition :

Definition 2 A point c of a set $A \subseteq R$ is said to be an interior point of A if for some $\delta > 0$, $]c - \delta, c + \delta[\subseteq A$.

The point $1 \in A = [0, 2]$ is an interior point, since $]1 - \delta, 1 + \delta[\subset [0, 2]$, if we choose $\delta = \frac{1}{2}$. But neither 0, nor 2 is an interior point of A.

The following theorem gives us a necessary condition for the existence of an extremum.

Theorem 1 Let f be a function which is derivable at an interior point c of its domain D. If f has an extremum at c, then $f'(c) = 0$.

Proof : Since f is derivable at c, $f'(c)$ exists. This means that $Lf'(c)$ and $Rf'(c)$ exist, and are equal. Suppose, first, that f has a maximum at c. This means.

$f(x) \leq f(c), \forall x \in]c - \delta, c + \delta[$, where δ is some positive number.

$\implies f(x) - f(c) \leq 0 \; \forall x \in]c - \delta, c + \delta[$

Now if $c - \delta < x < c$, then $x - c < 0$ and $\frac{f(x) - f(c)}{x - c} \geq 0$

Similarly, if $c < x < c + \delta$, then $x - c > 0$, and $\frac{f(x) - f(c)}{x - c} \leq 0$

Therefore

$$Lf'(c) = \lim_{x \to c^-} \left[\frac{f(x) - f(c)}{x - c} \right] \geq 0, \text{ and}$$

$$Rf'(c) = \lim_{x \to c^+} \left[\frac{f(x) - f(c)}{x - c} \right] \leq 0$$

But we know that $Lf'(c) = Rf'(c)$. Hence the above inequalities yield $Lf'(c) = Rf'(c) = 0$, that is, $f'(c) = 0$.

Proceeding exactly as above, we can prove that $f'(c) = 0$ even when f has a minimum at c (see E 2)).

Remark 2 The condition $f'(c) = 0$ is only a necessary condition for f to have an extremum at c. This means that if f has an extremum at c, then we must necessarily have $f'(c) = 0$. But it is by no means a sufficient condition. In other words, if we are given that $f'(c) = 0$, this information is not sufficient for us to conclude that f has an extremum at c. That is, a function may not have an extreme value at c even though $f'(c)$ is zero. For example, take the function $f(x) = x^3$ shown in Fig. 6. We know that $f'(0) = 0$. Now, $f(0) = 0$, whereas $f(x) < 0$ for $-\delta < x < 0$, and $f(x) > 0$ for $0 < x < \delta$, whatever positive number δ may be. Thus there is no δ for which either $f(x) \leq f(0)$, for all $x \in]-\delta, \delta[$ or $f(x) \geq f(0)$, for all $x \in]-\delta, \delta[$.

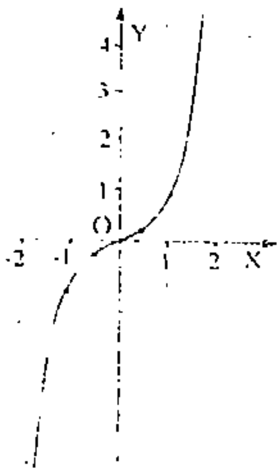


Fig. 6

This indicates that f has neither a maximum, nor a minimum at $x = 0$, even though $f'(0) = 0$.

Remark 3 The condition $f'(c) = 0$ applies in the case when $f'(c)$ exists. But a function may have an extremum at $x = c$ even though it is not derivable at c. For instance, the function f of Example 4 is not derivable at $x = 1$. Yet it has a minimum there. Similarly, function $f(x) = -|x|$ is not derivable at $x = 0$ but has a maximum there. Can you point out another function from the examples above, which is not derivable but has an extremum?

From this discussion we arrive at the following corollary:

Corollary 1 If a function f has an extreme value at a point $x = c$, then one of the following conditions is satisfied :

i) f is not derivable at c.

ii) f is derivable at c, and $f'(c) = 0$.

Definition 3 A point at which either $f'(x)$ does not exist or is zero is known as a critical point.

If you study Definition 3 and Cor. 1 carefully, you will realise that any extreme point is a critical point. But the converse is not true. That is, a critical point need not be an extreme point. You will find an example to illustrate this in Remark 2 above.

Example 5 Consider the function $f(x) = x^{2/3}$ for all $x \in \mathbb{R}$.

$f'(x) = \frac{2}{3} x^{-1/3}$. Thus, this function is not derivable at $x = 0$.

This means that f has a critical point at $x = 0$.
Further, at other points f' exists, but
$f'(x) = (2/3)\, x^{-1/3} \neq 0$.

Hence, the function has only one critical point, namely 0. For every $\delta > 0$, and every x in $]-\delta, \delta[$, $f(x) = x^{2/3} = (x^{1/3})^2 \geq 0$ and $f(0) = 0$. Thus $x = 0$ is a minimum point and $f(0) = 0$ is the minimum value (also see Fig. 7).

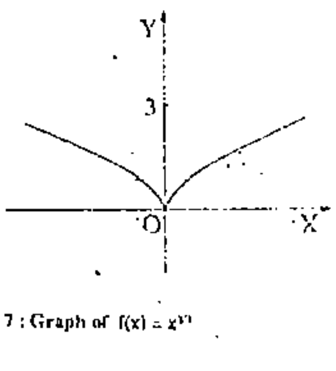


Fig. 7 : Graph of $f(x) = x^{2/3}$

Example 6 A farmer has a certain length of fencing wire. He wants to make a fence in a rectangular shape to keep his goats. Now what dimensions should he choose to ensure maximum enclosed area?

It is clear that in order to get a maximum enclosed area, the farmer should use the whole length of wire. Suppose it is 4k metres. Then 4k is the perimeter of the enclosure. Thus if x and y are the length and the breadth, respectively, of the enclosure (in metres), then $2(x + y) = 4k$, or $y = 2k - x$, $0 < x < 2k$.

Thus, we can regard the enclosed area xy (in m^2) as a function f of its length x. Thus, we have the function f defined as
$f(x) = x(2k - x)$, for all $x > 0$ (length is always positive). Our task is to find that value of x, for which f(x) becomes maximum. Now f(x) is a polynomial in x, and therefore f is derivable for all x. Hence its critical points (among which we should seek the extreme values) are obtained from the equation $f'(x) = 0$, which in this case gives $2k - 2x = 0$, that is $x = k$.

This means that the function $f(x) = x(2k - x)$ has a critical point at $x = k$.

Let us see whether it has an extremum at $x = k$.

Now, $f(k) = k^2$, and

$$f(x) = 2kx - x^2 = k^2 - (x - k)^2 \qquad \text{..... (1)}$$

Since there is only one maximum value, namely f(k), of f(x), $f(k) \geq f(x)$ for all x s.t. $0 < x < 2k$.

It is obvious from (1) that for values of x less than or greater than k, $(x - k)^2 > 0$ and therefore, $f(x) < k^2 = f(k)$. Hence, for any $\delta > 0$, $f(x) \leq f(k)$ for all $x \in]k - \delta, k + \delta[$. This means f(k) is a local maximum. But since there is only one extreme point, f(k) is the maximum value of the area. Now, when $x = k$, $y = 2k - x = k$. So, the farmer should have a square fence made, taking the entire length of his wire.

In this example it was easy to compare the neighbouring function values with f(k). Are you wondering how you would tackle this problem when the functions under consideration would not be so nice? There is no need for speculation. We have sufficient conditions which tell us whether f(x) is an extreme value or not, and if it is, whether it is a minimum or a maximum value, without actually having to compare values. Section 6 is, in fact, devoted to such explorations. Before that, in the next section we shall discuss the mean value theorems. But now it's time to do some exercises.

E E 2) Prove that $f'(c) = 0$ if f has a minimum at c.

E 3) Find the critical points of each of the following functions.

a) $f(x) = (x - 3)(x - 5), \forall x \in R.$

b) $f(x) = x^3 + 13x^2 + 5x + 7, \forall x \in R.$

c) $f(x) = \sin x + 3, \forall x \in R.$

d) $f(x) = e^x, \forall x \in R.$

e) $f(x) = 2|x|, \forall x \in R.$

f) $f(x) = |x| + 2, \forall x \in R.$

g) $f(x) = |x| + |x - 1|, \forall x \in R.$

h) $f(x) = x + 1/x, x > 0$

7.3 MEAN VALUE THEOREMS

In this section we shall study the mean value theorems. These theorems have proved to be very handy tools in proving other theorems not only in calculus, but also in other branches of mathematics, such as Numerical Analysis. Their importance lies in their wide applicability and tremendous usefulness.

We shall first consider Rolle's Theorem, which is a special case of Lagrange's mean value theorem. We shall not attempt the proofs of these theorems here, but you will agree that both are intuitively obvious. We shall discuss their geometrical significance and illustrate their usefulness through some examples.

7.3.1 Rolle's Theorem

Rolle's Theorem was not actually proved by Rolle. He had only stated it as a remark. In fact, Michel Rolle (1652-1719) was known to be a critic of the newly founded theory of

Newton and Leibniz. It is ironical, then, that one of the most important theorems of this theory is known after him. Now let us see what this theorem is.

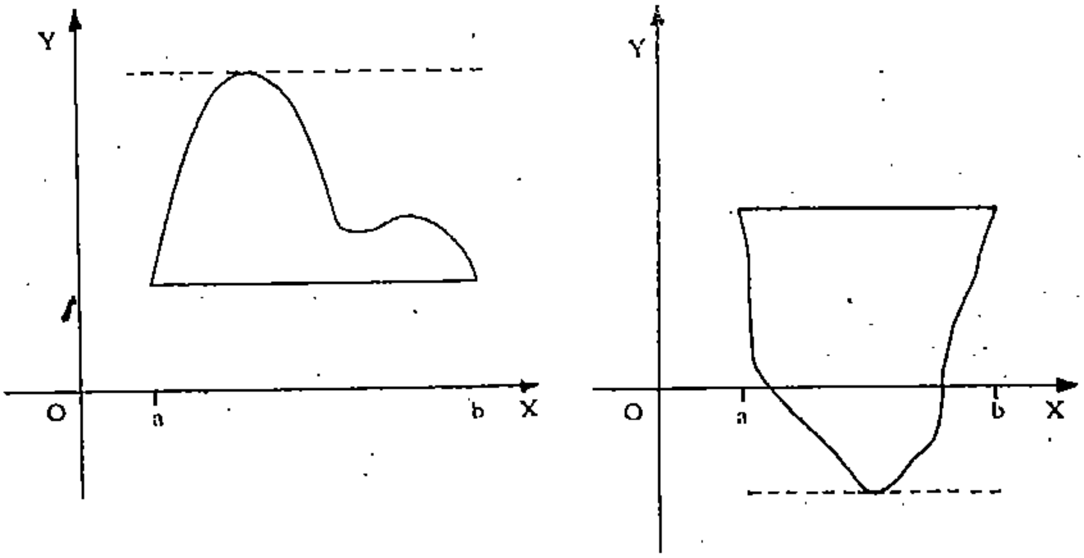


Fig. 8

In Fig. 8 we see the graphs of two continuous functions defined on the closed interval [a,b].

Here we observe the following features common to both of them.

Rough Statement	Precise Statement
1 The curve is drawn without breaks or gaps.	The function f is continuous on [a,b].
2 There are no corners in the curve, except possibly at the end points.	The function is differentiable in the open interval]a,b[.
3 The two end points of the curve lie on the same horizontal line.	f(a) = f(b)
4 The curve admits a horizontal tangent (drawn as a dotted line) at some point.	f′(c) = 0 for some c in]a, b[.

The line joining the two end points may be imagined to be pushed upward or downward, keeping it always horizontal, and keeping the curve unmoved. Then there is a position, shown by the dotted line, where it touches the curve. This makes us believe that the fourth property holds for all the functions satisfying the first three properties. This is what Rolle's Theorem states.

Theorem 2 (Rolle's Theorem) Let f be a function continuous on the closed interval [a, b] and differentiable in the open interval]a, b[. Further, let f(a) = f(b). Then there is some c in]a, b[such that $f'(c) = 0$.

We give some examples below to illustrate this theorem.

Example 7 Consider $f(x) = \sin x$ on the interval $[0, 2\pi]$. All the assumptions of Rolle's theorem are satisfied here. $f(0) = 0 = f(2\pi)$

Therefore according to Rolle's theorem, there should exist c in $]0, 2\pi[$ such that $f'(c) = 0$. Here $f'(c) = \cos c$.

Can we find an element c such that $\cos c = 0$?

Yes. In fact there are two such points c in $]0, 2\pi[$, namely $\pi/2$ and $3\pi/2$.

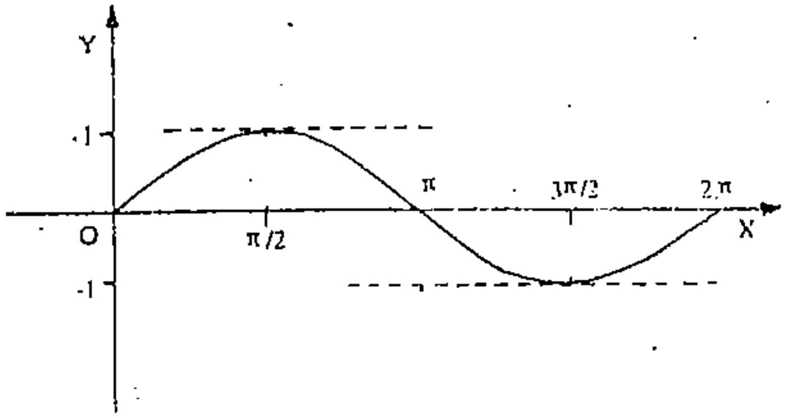


Fig. 9

At $\pi/2$, the function sin x attains its maximum value.
At $3\pi/2$, the function sin x attains its minimum value.
Both these belong to the interval $]0, 2\pi[$.

Rolle's theorem asserts that there is at least one c in]a, b[such that $f'(c) = 0$. Example 7 shows us that there may be more than one points in]a, b[, at which $f'(x) = 0$.

In Rolle's theorem, a function f on [a, b] has to satisfy three conditions.

i) f is continuous on [a, b]

ii) f is differentiable on]a, b[

iii) $f(a) = f(b)$

Now, we shall see through some examples that each of these conditions is essential. We cannot drop any one of them and still prove the theorem.

Example 8 Let $f(x) = x - [x]$ = fractional part of x, be defined on [0, 1]. This can also be described as

$$f(x) = \begin{cases} x \text{ if } 0 \le x < 1. \\ 0 \text{ if } x = 1. \end{cases}$$

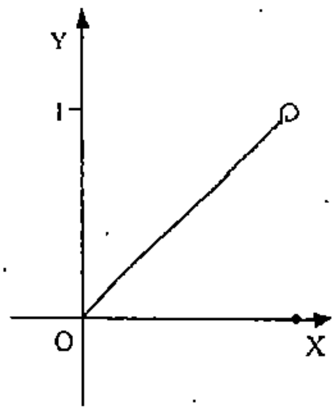


Fig. 10

Here $f(0) = f(1) = 0$. f is differentiable in the open interval]0, 1[. Thus, two of the three conditions of Rolle's theorem are satisfied by f. The derivative of f is 1 at every point of]0, 1[. There is no point of]0, 1[where the derivative is zero. What happens to Rolle's Theorem in this example? Obviously, its conclusion does not hold here.

The reason is that f is not continuous on the closed interval [0,1], since it fails to be continuous at 1.

In the next example, we see that the assumption of differentiability in]a, b[cannot be omitted.

Example 9 Consider $f(x) = |x|$ on $[-1, 1]$. There is no c in $]-1, 1[$ such that $f'(c) = 0$. Actual computation shows that

$$f' = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ \text{does not exist at } x = 0. \end{cases}$$

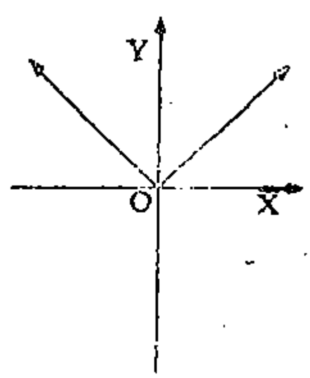


Fig. 11

f is continuous on $[-1, 1]$.
Also, $f(-1) = f(1)$.
But f is not differentiable in $]-1, 1[$.
Our next example shows that the assumption $f(a) = f(b)$ is essential in Rolle's Theorem.

Example 10 Let $f(x) = x^3$ on [0, 1]. Then f is continuous on [0, 1], and is differentiable in]0, 1[. But $f(0) \ne f(1)$.

In this case $f'(x) = 3x^2 \neq 0$ for any $x \in]0, 1[$. Thus, we see that the conclusion of Rolle's theorem may not hold when $f(a) \neq f(b)$.

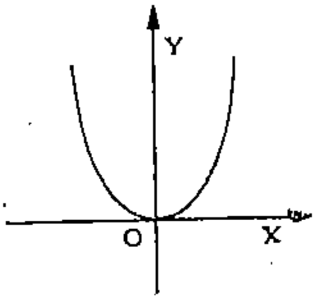


Fig. 12

Lastly, we give an example where Rolle's Theorem is applicable and yields a unique c.

Example 11 Let $f(x) = x^2$ on $[-1, 1]$. Then $f'(x) = 2x$.
Here all the three conditions of Rolle's Theorem are satisfied. There is only one c, namely $c = 0$, such that $f'(c) = 0$.

You will now be able to solve these exercises.

E E 4) Can Rolle's Theorem be applied to each of the following functions? Find 'c' in case it can be applied.

a) $y = \sin^2 x$ on the interval $[0, \pi]$.

b) $f(x) = x^2 + 1$ on $[-2, 2]$.

c) $f(x) = x^3 + x$ on $[0, 1]$.

d) $f(x) = \sin x + \cos x$ on $[0, \pi/2]$.

e) $f(x) = \sin x - \cos x$ on $[0, 2\pi]$.

E E 5) Consider the function $f(x) = x^2 - 3x + 2$. Prove that $f(-1) = f(4)$. Find a point c between -1 and 4 such that the derivative of f vanishes at c. Is this point the midpoint of -1 and 4?

E 6) For the same function $f(x) = x^2 - 3x + 2$, verify Rolle's Theorem on the interval $[1, 2]$.

E 7) Let $f(x) = ax^2 + bx + c$ be the given function. If p and q are two real numbers such that $f(p) = f(q)$, prove that $f'\left[\frac{p + q}{2}\right] = 0$.

E 8) Consider the curve $y = ax^2 + bx + c$. Let x_0 be the unique real number such that the tangent at (x_0, y_0) to this curve is horizontal. Prove that the function y is one-one on the interval $[x_0, \infty[$ (Hint : If $f(p) = f(q)$, apply Rolle's theorem in the interval $[p, q]$).

E

E 9) Let I be an open interval of R. Let $f : I \to R$ be a differentiable function such that f' does not vanish on I. Prove that f is one-one on I.

7.3.2 Lagrange's Mean Value Theorem

Now we shall discuss the mean value theorem. It was proved by Joseph Louis Lagrange, a towering mathematician of the eighteenth century.

We have already mentioned that Rolle's theorem is a special case of the mean value theorem. Let us recall the statement of Rolle's Theorem in the following form.

Let f be a continuous function on the closed interval [a, b], and differentiable in the open interval]a,b[. The graph of f is a curve in the plane. If the endpoints of this curve lie in the same horizontal line, (that is, $f(a) = f(b)$) there is a point c on the curve where the tangent to the curve is horizontal ($f'(c) = 0$)..

The last sentence can be restated as follows.

If the endpoints of this curve lie in the same horizontal line, there is a point on the curve, where the tangent to the curve is parallel to the line joining its endpoints.

The mean value theorem asserts the same conclusion, even without the assumption of horizontality of the line joining the endpoints of the curve. Fig. 13 illustrates this difference. Here P and Q are the end points of the curve. The line PQ is horizontal in Fig. 13(a), but not in Fig. 13(b). But in both the cases the point R on the curve has the property that the tangent to the curve at R is parallel to the line PQ. The number c is the x-coordinate of R.

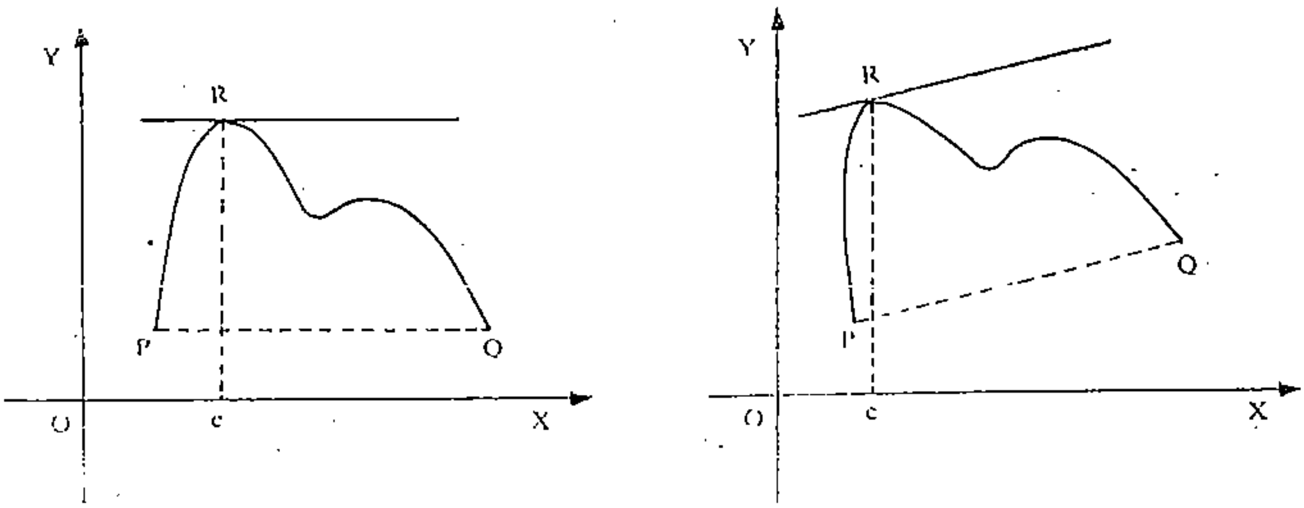


Fig. 13

Fig. 13 (a) illustrates Rolle's theorem, whereas Fig. 13(b) illustrates Lagrange's mean value theorem.

The two end points of the curve are (a, f(a)) and (b, f(b)). The line joining these two points has the slope $[f(b) - f(a)]/(b - a)$. Any line parallel to this line will also have the same slope. Therefore, the conclusion of the mean value theorem is
$f'(c) = [f(b) - f(a)]/(b - a)$ for some $a < c < b$.

This is because, we already know that $f'(c)$ is the slope of the tangent to the curve at (c, f(c)). Now we are ready to give the precise statement of the theorem.

Theorem 3 (Lagrange's Mean Value Theorem) Let f be a continuous function on a closed interval [a, b]. Let f be differentiable in the open interval]a, b[. Then there is a point c in the open interval]a, b[such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Rolle's Theorem has three assumptions : a continuity assumption, a differentiability assumption, and the assumption $f(a) = f(b)$.

The mean value theorem has only two assumptions. These are the same as the first two assumptions of Rolle's Theorem.

Suppose in addition to the two assumptions of the mean value theorem, $f(a) = f(b)$ also holds. Then what does the mean value theorem yield? It says that
$f'(c) = \frac{f(b) - f(a)}{b - a}$ for some $a < c < b$. But $f(b) - f(a) = 0$.

Therefore, we get $f'(c) = 0$ for some $a < c < b$.

This is the same as the conclusion of Rolle's theorem. This proves our contention that Rolle's theorem can be deduced from the mean value theorem.

But why the name mean value theorem? What is the mean value here?
f(a) is the initial value of f.
f(b) is the final value of f.

Therefore $f(b) - f(a)$ is the total change in the value of f. This change has occurred when the x-coordinate has changed from a to b. For a change of $b - a$ in the domain, there is a change of $f(b) - f(a)$ in the value of f. Therefore, the mean value, that is, the average value of the rate of change is $[f(b) - f(a)]/(b - a)$. The mean value theorem asserts that this average value of the rate of change of f is assumed at some point c by derivative f'.

We shall illustrate the same thing by means of an example. Consider a car moving from time a to time b; let f(t) be the position of the car at time t. Then the average speed of the car is
distance/time $= [f(b) - f(a)]/(b - a)$

According to mean value theorem, the speedometer of the car would have shown this $[f(b) - f(a)]/(b - a)$ at some time between a and b. For instance, if the car has travelled 100 kms. in two hours, at some point of time, its speed would have been actually 50 kmph. (which is its average velocity over the span of two hours).

Example 12 Let us verify the truth of Lagrange's mean value theorem for the function $f(x) = x^2 - 2x$ on the interval [1, 2].

This is a polynomial function. Therefore it is continuous on [1,2] and differentiable in]1, 2[. (We have seen this in Units 2 and 3). Here

$a = 1, b = 2$
$f(a) = 1 - 2 = -1$
$f(b) = 2^2 - 2 \times 2 = 0$
$$\frac{f(b) - f(a)}{b - a} = \frac{0 - (-1)}{2 - 1} = 1$$

We want to check that $f'(c) = 1$ for some c such that $1 < c < 2$.
Now $f'(x) = 2x - 2$. For what value of x will it be 1?

Now, $2x - 2 = 1$ when $x = 3/2$ and $3/2 \in]1, 2[$. Thus, we see that

$$f'(3/2) = \frac{f(2) - f(1)}{2 - 1}.$$

Now consider the function $f : [a, b] \to \mathbb{R}$ which satisfies the assumptions of mean value theorem. Let p and q be any two points such that $a \le p < q \le b$. Is there some c between p and q such that

$f'(c) = [f(q) - f(p)]/(q - p)$? To answer this, consider the restriction of f to the interval [p, q]. It satisfies the assumptions of the mean-value-theorem. Therefore such a point c exists.

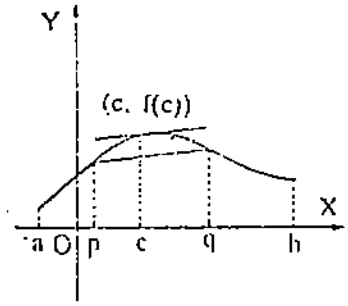


Fig. 14

This result can be geometrically interpreted as follows. (p, f(p)) and (q, f(q)) are two points on the curve $y = f(x)$.

The line joining them is called a chord of the curve. $\frac{f(q) - f(p)}{q - p}$ is the slope of this chord. What we have shown is that the slope of this chord is the same as the slope of the tangent at the point (c, f(c)). This means, that the tangent at (c, f(c)) is parallel to the chord (see Fig. 14). Thus, for any chord of the curve, there is a point on the curve where the tangent is parallel to the chord.

Example 13 i) Let us find the point c in $]-\pi/4, \pi/4[$ such that the tangent to $f(x) = f(x) = \cos x$ at (c, f(c)) is parallel to the chord joining $(-\pi/4, f(-\pi/4))$ and $(\pi/4, f(\pi/4))$.

ii) We shall prove further that for the same c, the tangent at (c, g (c)) to the curve $g(x) = \cos x + x^2 + x$ is parallel to the chord joining $(-\pi/4, g(-\pi/4))$ and $(\pi/4, g(\pi/4))$.

i) The slope of the chord joining $(-\pi/4, f(-\pi/4))$ and $(\pi/4, f(\pi/4))$ is

$$\frac{f(\pi/4) - f(-\pi/4)}{\pi/4 - (-\pi/4)} = \frac{1/\sqrt{2} - 1/\sqrt{2}}{\pi/2} = 0.$$

Therefore we seek c such that $f'(c) = 0$. We have $f'(x) = -\sin x$.
The only point in $]-\pi/4, \pi/4[$ where this vanishes is at $c = 0$.
The corresponding point on the curve is $(0, f(0)) = (0,1)$.

ii) $g(-\pi/4) = (1/\sqrt{2}) + (\pi^2/16) - (\pi/4)$

$g(\pi/4) = (1/\sqrt{2}) + (\pi^2/16) + (\pi/4)$.

The slope of the chord joining $((-\pi/4), g(-\pi/4))$ and $(\pi/4, g(\pi/4))$ is

$$\frac{g(\pi/4) - g(-\pi/4)}{\pi/4 - (-\pi/4)} = \frac{(\pi/4) + (\pi/4)}{(\pi/4) + (\pi/4)} = 1.$$

When $c = 0$, we want to prove that the tangent at (c, g(c)) to the curve g(x) also has the same slope 1. In other words, we must prove that $g'(0) = 1$.

Now, $g'(x) = -\sin x + 2x + 1$.
$\therefore \quad g'(0) = -0 + 0 + 1 = 1$.

This proves that, for both the functions f(x) and g(x) over $]-\pi/4, \pi/4[$, it is the same point c where the conclusion of the mean value theorem holds.

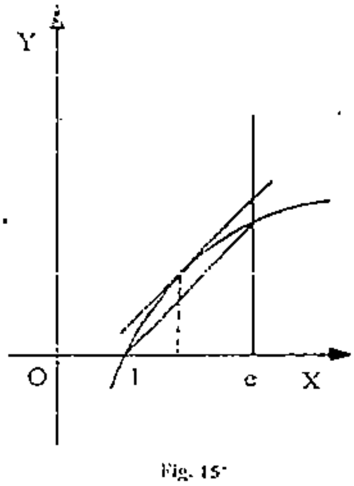


Fig. 15

Example 14 For the curve $y = \ln x$, suppose we want to find a point on the curve where the tangent is parallel to the chord joining the points (1, 0) and (e, 1).

Since $\ln 1 = 0$ and $\ln e = 1$, these two points (1, 0) and (e, 1) lie on the curve $y = \ln x$. Consider this function on the closed interval [1, e] (see Fig. 15). It is continuous there. It is also differentiable on]1, e[.

Therefore by the mean value theorem, there is a point c between 1 and e such that the tangent at (c, ln c) is parallel to the chord joining (1, 0) and (e, 1). We have to find this point. Now, $y' = 1/x$.
Its value at c is $1/c$.

The required point is given by

$$\frac{1}{c} = \frac{\ln e - \ln 1}{e - 1} = \frac{1 - 0}{e - 1} = \frac{1}{e - 1}$$

$\therefore \quad c = e - 1$

The required point on the curve is $(e - 1, \ln(e - 1))$.

Remark 4 Let $f : [a, b] \to R$ satisfy the assumptions of the mean value theorem. Then $\exists\ \theta, 0 < \theta < 1$, such that $f(b) = f(a) + (b - a) f'(a + \theta(b - a))$.

This is because any point c between a and b is of the form $a + \theta(b - a)$ for some $0 < \theta < 1$. Note that $a = a + 0(b - a)$ and $b = a + 1(b - a)$.

Are you ready for these exercises?

E 10) Verify the mean value theorem for $f(x) = x^2 + 1$ on the following intervals

a) $[-1, 1]$ b) $[-1, 2]$

E 11) Verify the mean value theorem on the interval $[0, 2]$ for the following functions

a) $f(x) = \sin \pi x$

b) $f(x) = 2x^2 + 3$

E E 12) a) Let $f(x) = x^3$ on $[0, 1]$. Find a point c in $]0, 1[$ as in the mean value theorem.

b) Let $f(x) = x^3$ on $[-1, 0]$. Find a point c in $]-1, 0[$ as in the mean value theorem.

c) Let $f(x) = x^3$ on $[-1, 1]$. Show that there are two points c in $]-1, 1[$ such that $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$

E E 13) Let f be a function on $[a, b]$ satisfying the assumptions of the mean value theorem. Let c be a point guaranteed by the mean value theorem. Prove that if

$g_1(x) = f(x) + 1$

and $g_2(x) = f(x) + x$ for all x in $[a, b]$, then the same point c satisfies

$$\frac{g_1(b) - g_1(a)}{b - a} = g'_1(c)$$

and $\frac{g_2(b) - g_2(a)}{b - a} = g'_2(c)$ also.

E 14) At what point is the tangent to the curve $y = x^n$ parallel to the chord from

a) (0, 0) to $(2, 2^n)$?

b) (0, 0) to (1, 1)?

Just as in the case of Rolle's theorem, there may be more than one points at which the tangents may be parallel to the chord joining the end points of a curve represented by a function which is continuous at every point in the closed interval and is differentiable at every point in the open interval (see Fig. 16).

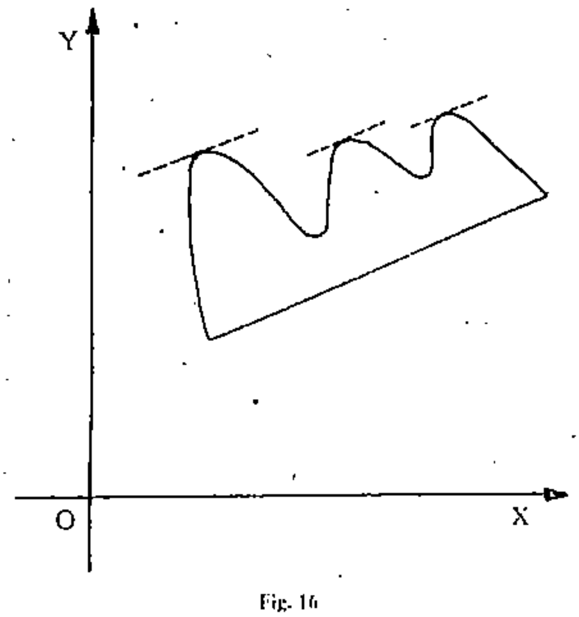


Fig. 16

Both Rolle's theorem and Lagrange's mean value theorem are existence theorems. They tell us that there exists at least one point where the tangent is parallel to the chord joining the end points. But they do not tell us how many such points are there, nor how to find these points. For example, consider the function

$f(x) = x^3 - \sin x$ on $[0, 5\pi]$

It satisfies the conditions of the mean value theorem. So there is at least one value c at which

$3c^2 - \cos c = 25\pi^2$

The mean value theorem assures us that the equation $3x^7 - \cos x = 25x^2$ has atleast one solution. c. But it does not enable us to find the value or values of c. You can study methods of solving such equations in the course in numerical analysis.

In the next section we shall see how the mean value theorem helps us to derive sufficient conditions for the existence of extreme points of a function.

7.4 SUFFICIENT CONDITIONS FOR THE EXISTENCE OF EXTREME POINTS

In Sec. 2 we have seen that a necessary condition for the existence of an extreme point of a given derivable function is that the derivative is zero at that point. We have also seen that the condition is not a sufficient one. In this section we shall discuss some tests which give sufficient conditions for the existence of extreme points. Lagrange's mean value theorem which we have studied in Sec. 3 is used in deriving these tests.

7.4.1 First Derivative Test

The following theorem gives a sufficient condition for a function f to have an extreme value at an interior point c of its domain. It also tells us whether the extreme value is a minimum or a maximum.

Theorem 4 (FIRST DERIVATIVE TEST) Let c be an interior point of the domain of a function f. Suppose f is derivable on $]c - \delta, c + \delta[$ for some $\delta > 0$ and that $f'(c) = 0$. Then

a) if $f'(x) > 0$ when $c - \delta < x < c$ and
$f'(x) < 0$ when $c < x < c + \delta$, then f has a maximum at c, and

b) if $f'(x) < 0$ when $c - \delta < x < c$
and $f'(x) > 0$ when $c < x < c + \delta$, then f has a minimum at c.

Proof a) We have to prove that f has a maximum at c. In other words, we have to show that $f(c) \geq f(x)$ for all x in some neighbourhood of c.

Now, let $x \in]c - \delta, c[$. Then f is differentiable on $]x, c[$ (in fact on $[x, c]$), and continuous on $[x, c]$ because differentiability at a point implies continuity there (Theorem 6, Unit 3). By Lagrange's theorem, there exists $a \in]x, c[$ such that

$\frac{f(c) - f(x)}{c - x} = f'(a)$. That is,

$$f(c) - f(x) = (c - x) f'(a). \quad \text{..... (1)}$$

Since $f'(x) > 0$ on $]c - \delta, c[$ and $a \in]x, c[\subset]c - \delta, c[$,
therefore, $f'(a) > 0$. Also, $c - x > 0$ since $x \in]c - \delta, c[$. Hence, from (1)

$$f(c) - f(x) > 0 \text{ or } f(c) > f(x). \quad \text{..... (2)}$$

Similarly, if $x \in]c, c + \delta[$, then by Lagrange's theorem, there exists a point $b \in]c, x[$ such that

$$f(x) - f(c) = (x - c) f'(b). \quad \text{.....(3)}$$

Since $f'(x) < 0$, when $x \in]c, c + \delta[$, $f'(b) < 0$. Further, since $x \in]c, c + \delta[$, $x - c > 0$.

(3) now gives

$$f(x) - f(c) < 0 \text{ or } f(c) > f(x). \quad \text{..... (4)}$$

Thus, it follows from (2) and (4) that whether $x \in]c - \delta, c[$ or to $]c, c + \delta[$, $f(c) > f(x)$. In other words, $f(c) \geq f(x) \ \forall x \in]c - \delta, c + \delta[$. Hence, f(x) is a maximum value of f and f has a maximum at c.

The proof of part (b) proceeds on similar lines. See if you can write it yourself.

E F 15) Prove part (b) of Theorem 4.

We can also interpret Theorem 4 as follows.

If while crossing a point, moving from left to right, the derivative changes sign from positive to negative, then that point must be a maximum. Similarly, while crossing a point, if the derivative changes sign from negative to positive, then that point must be a minimum.

Remark 5 From Theorem 4 we can deduce that when $f'(x) > 0$ in an interval, the function f is strictly increasing in that interval. Similarly, $f'(x) < 0$ in an interval would imply that f is a strictly decreasing function in that interval. $f' = 0$ in an interval, likewise, means that the function is a constant function in that interval. We shall talk about this in Block 4.

You should note that the conditions stated in Theorem 4 are sufficient and are by no means necessary for the existence of extreme points. Thus, a function may have an extreme value at c even though the conditions given in the theorem are not satisfied as shown in the following example.

Example 17 Let $f(x) = x^2[5 + \sin(1/x)]$, $x \neq 0$, and $f(0) = 0$.
Then $f(x) > 0$ if $x \neq 0$, as $x^2 > 0$ and $5 + \sin(1/x) > 0$ since $\sin(1/x) \geq -1$. Thus f has a minimum at $x = 0$. $f'(x) = 2x[5 + \sin(1/x)] - \cos(1/x)\ \forall x \neq 0$, and

$$
\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \\
&= \lim_{h \to 0} \frac{h^2[5 + \sin(1/h)]}{h} \\
&= \lim_{h \to 0} h[(5 + \sin(1/h)] \\
&= 0 \text{ since } 4 \leq 5 + \sin(1/h) \leq 6
\end{aligned}
$$

Thus, $f'(x)$ exists for all x. However, neither in the interval $]-\delta, 0[$, nor in $]0, \delta[$ does $f'(x)$ keep the same sign, no matter how small δ is (you may check this yourself). Thus, the question of $f'(x)$ being positive or negative on two sides of 0 does not arise.

Example 18 Among all rectangles having a given area, is there one which makes the perimeter a minimum?

Suppose that the length and the breadth of the rectangle are x and y, respectively. Then $x > 0$, $y > 0$. The area xy being constant, k^2 say, we have $y = k^2 x^{-1}$. Then, the perimeter of this rectangle
= 2 (length + breadth)
= $2(x + k^2 x^{-1})$, $x > 0$. This is a function of x alone. Let us denote it by f(x). Thus,
$f(x) = 2(x + k^2 x^{-1})$, $x > 0$.

We want that value of x which makes f(x) a minimum. The function f is derivable at all points of its domain. The derivative is given by

$f'(x) = 2[1 - (k^2/x^2)]$

Now, $f'(x) = 0 \Rightarrow x^2 = k^2 \Rightarrow x = \pm k$. If we assume that $k > 0$, then $-k$ is not in the domain of f. Thus the only critical point is $x = k$. Let δ be any positive number such that $k - \delta > 0$. Then f is derivable at all points of $]k - \delta, k + \delta[$.

i) $x \in]k - \delta, k[\Rightarrow x < k,$
$\Rightarrow x^2 < k^2,$
$\Rightarrow k^2/x^2 > 1,$
$\Rightarrow f'(x) < 0$

ii) $x \in]k, k + \delta[\Rightarrow x > k,$
$\Rightarrow k^2/x^2 < 1,$
$\Rightarrow f'(x) > 0.$

Hence by Theorem 4, f has a minimum at $x = k$. Also, when $x = k$,
$y = k^2 x^{-1} = k.$

This means that a square shape yields a minimum perimeter among all rectangles with a fixed area.

You should be able to solve the following exercise now.

E E 16) Find all possible extreme values of each of the following functions by applying the first derivative test.

a) $f(x) = x^5 - 5x^4 + 5x^3 - 1$ for all $x \in R$
[It may be helpful to factorize $f'(x)$]

b) $f(x) = 2x^4 + 8x^3 - 4x^2 - 24x + 15$ for all $x \in R$

c) $f(x) = (x - 1)^2 (x + 1)$ for all $x \in R$

7.4.2 Second Derivative Test

We now investigate another condition which, if satisfied, does away with the need to examine the sign of $f'(x)$ in $]c-\delta, c[$ and $]c, c+\delta[$ as in the first derivative test. This condition also is only sufficient, but very functional and hence, useful.

Theorem 7 (Second Derivative Test) Let f be derivable in $]c-\delta, c+\delta[$ for some $\delta > 0$ and suppose $f'(c) = 0$.
Then

a) f has a maximum at $x = c$ provided $f''(c)$ exists and is negative.

b) f has a minimum at $x = c$ provided $f''(c)$ exists and is positive.

Proof: a) Since $f''(c) < 0$, f' is a strictly decreasing function in the neighbourhood of c (see Remark 5). Thus there exists an $\varepsilon > 0$ (and we may take it smaller than δ) such that
$f'(x) > f'(c)$ when $x \in]c-\varepsilon, c[$ (since $x < c$)
and $f'(x) < f'(c)$ when $x \in]c, c+\varepsilon[$ (since, here, $x > c$)

Since $f'(c) = 0$, this means that $f'(x) > 0$ when $x \in]c-\varepsilon, c[$ and $f'(x) < 0$ when $x \in]c, c+\varepsilon[$.

By the first derivative test it follows that f has a maximum at c. The proof for (b) follows along similar lines.

Remark 6 You must have observed that this theorem says nothing about the case when $f''(c)$ is also zero. In this case the function may have a maximum or a minimum value or neither as the following examples show.

i) $f(x) = -x^4$, for all $x \in \mathbf{R}$.
Here $f'(0) = 0 = f''(0)$, but the function has a maximum at 0. (see Fig. 17(a)).

ii) $f(x) = x^4$, for all $x \in \mathbf{R}$.
Here $f'(0) = 0 = f''(0)$, but the function has a minimum at 0. (see Fig. 17(b)).

iii) $f(x) = x^3$, for all $x \in \mathbf{R}$.

Here $f'(0) = 0 = f''(0)$ and the function has neither a maximum nor a minimum at 0 (see Fig. 17(c)). Thus, the first derivative test does have some merit.

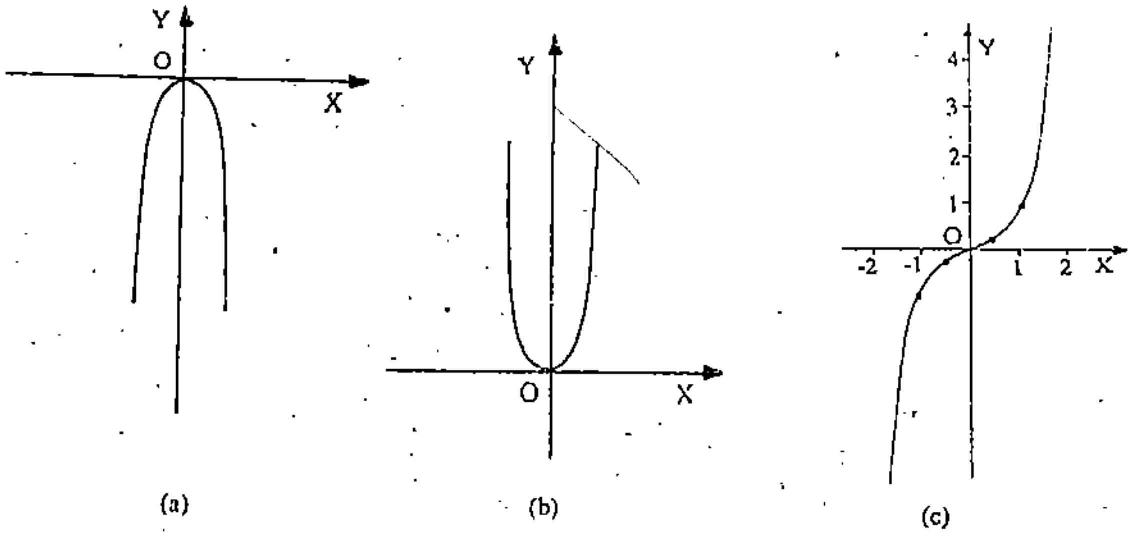


Fig. 17

Example 19 Let us find the extreme values of the function f defined by
$f(x) = 2x + 3/x$, for all $x \neq 0$
Here, $f'(x) = 2 - 3/x^2$, and therefore,
$$f'(x) = 0 \implies x = \pm\sqrt{3/2}$$
Also, $f''(x) = 6x^{-3}$. This means
$$f''(\sqrt{3/2}) > 0 \text{ and } f''(-\sqrt{3/2}) < 0.$$

Since, $f''(\sqrt{3/2}) > 0 > f''(-\sqrt{3/2})$, it follows from the second derivative test that f has a minimum at $\sqrt{3/2}$ and a maximum at $-\sqrt{3/2}$. The minimum value is $f(\sqrt{3/2}) = 2\sqrt{6}$, and the maximum value is $f(-\sqrt{3/2}) = -2\sqrt{6}$.

Example 20 From each corner of a square paper of side 24 cm, suppose we remove a square of side x cm and fold the edges upward to form an open box. Let us try to find that value of x which will give us a box with maximum capacity.

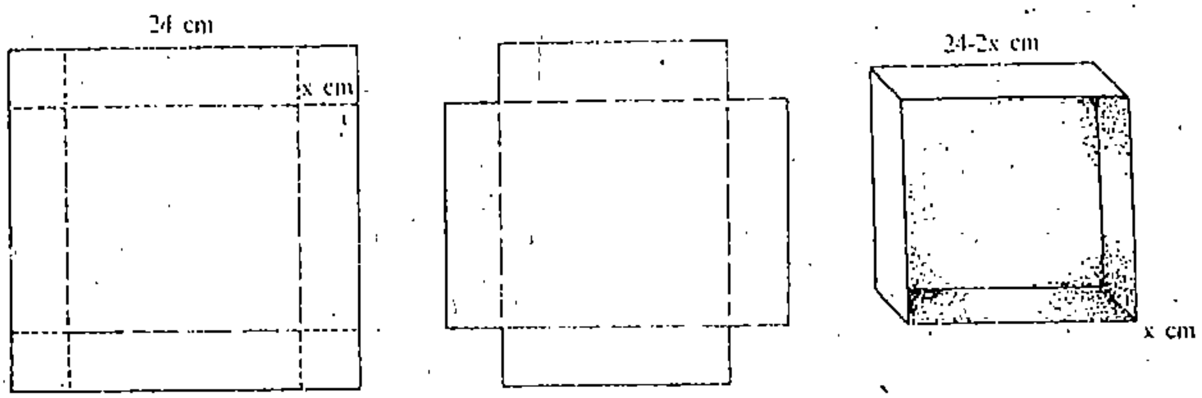


Fig. 18

Clearly, $0 < x < 12$ for a box to be formed. Also, the box thus formed has dimensions $(24 - 2x)$, $(24 - 2x)$ and x (see Fig. 18).
The volume f(x) is a function of x given by

$$f(x) = (24 - 2x)^2 x, \quad 0 < x < 12.$$
$$= 4x^3 - 96x^2 + 24^2x.$$
$$f'(x) = 12x^2 - 192x + 24^2 = 12(x - 4)(x - 12)$$

Now, $f'(x) = 0 \Rightarrow x = 12$ or $x = 4$.
Since 12 is not in the domain of our function f, 4 is the only critical point. Also,
$f''(x) = 24x - 192$.
Therefore, $f''(4) = 96 - 192 < 0$.
Hence $x = 4$ is a maximum point of f. The maximum value f(4) of f (that is, the maximum capacity of the box) is 1024 cm^3.

Are you surprised that the box is not a cube for maximum capacity? But, had it been a cube, four squares each of side 8 cm (the removed portions) would have been wasted, whereas, now four squares each of side only 4 cm have been thrown away. There had to be a compromise between the waste material and making the box as near a cube as possible! Moral : Mathematics does not fail even though intuition may!

Here are some exercises for you to solve.

E E 17) Find the extreme points for each of the following functions. Point out which of them are maximum, which are minimum and which are neither. Also find the extreme values of f.

a) $f(x) = x^2, x \in R$

b) $f(x) = -x^3, x \in R$

c) $f(x) = 3x^2 + 7x + 1, x \in R$

d) $f(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$, where $x \in R$ and each a_i is positive. [Do not get bogged down by the degree of f(x)!]

e) $f(x) = x/(x^2 + 1), 0 < x < \infty$

(Hint : A maximum point of f will be the minimum point of 1/f and vice versa).

E 18) Show that $\pi/3$ is a critical point of f, where
$f(x) = \sin x\,(1 + \cos x)$, $x \in \mathbb{R}$. Does f have a maximum or a minimum at this point?

E 19) Show that the rectangle of maximum area which can be inscribed in a circle, is a square.
[Notice that the diagonals of the rectangle must be the diameters of the circle].

E E 20) A man wants a name-plate with display area equal to $48cm^2$ bordered by a white strip 2 cm along top and bottom and 1 cm along each of the two remaining sides. What dimensions should the plate have so that the total area of the plate is a minimum?

7.5 MORE INFORMATION FROM THE SECOND DERIVATIVE

In the previous section we used the sign of the second derivative at a critical point to discover whether the function has a maximum or a minimum at that point. For drawing the necessary conclusion we regarded f'' as the derivative of the function f', and then applied the first derivative test. We shall now use the monotonicity considerations of f' to draw another conclusion, which would give us an idea about the shape of the graph of the function f. Given that a function f increases in [a, b], its graph may be anything like (a), (b) or (c) in Fig. 19

In this section we shall use the second derivative to determine the type of graph the given function f has. We shall assume that the function is twice differentiable in its domain.

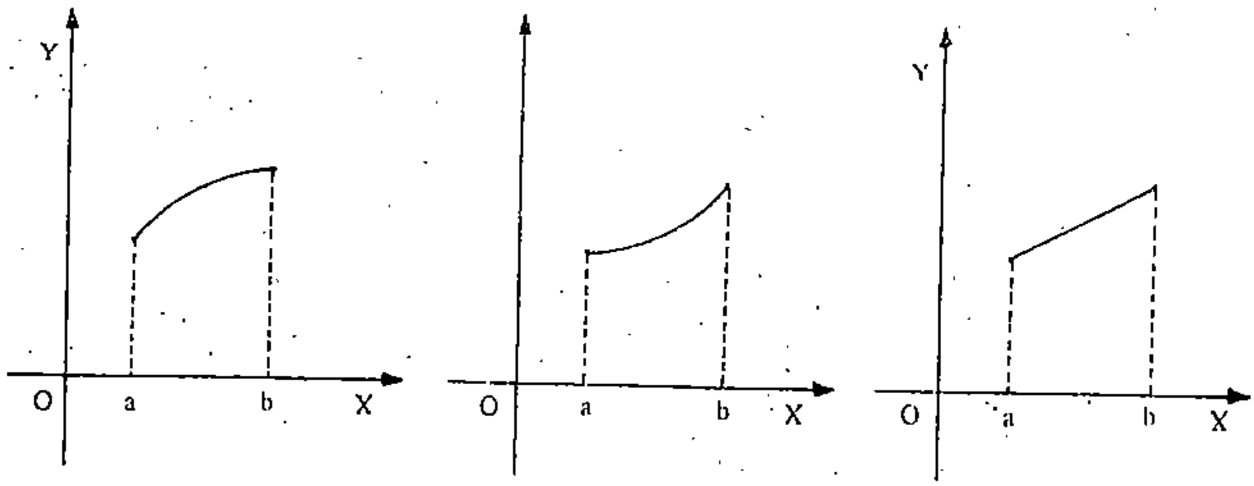
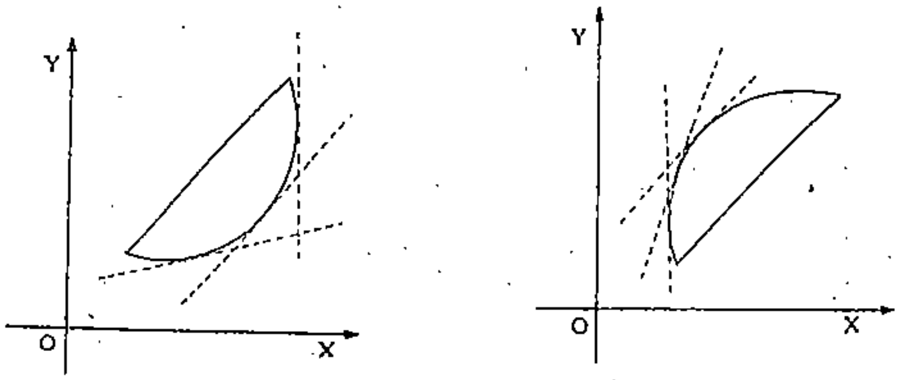


Fig. 19

7.5.1 Concavity/Convexity

If $f'' > 0$ in some interval $[a, b]$, then f' increases in this interval (Remark 5 applied to f'). In other words, as x increases, the slope of the tangent to the graph at $(x, f(x))$ increases, so that the tangent turns counter-clockwise. This results in the graph bending upward or bulging downward. Such graphs and functions are known as convex. We also use the terms concave upward or convex downward. Such graphs lie below their chord (line-segment joining the points on graph which correspond to $x = a$ and $x = b$) and above their tangents. See Fig. 20(a).

Similarly if $f'' < 0$ in $[a, b]$ so that f' decreases, and hence slope decreases, the tangent turns clockwise. Such functions are known as concave (equivalently, concave downward and convex upward). The graph in this case lies below the tangents and above the chord. (see Fig. 20(b)).



(a) (b)

Fig. 20

We say that a function f is concave (convex) at a point, if it is concave (convex) in a neighbourhood of that point.

Remark 7 : i) The concavity is towards the chord. If the chord lies above the graph, the graph is concave upward. When the chord lies below the graph, the graph is concave downward.

ii) Every convex function is of the form $-f$, where f is concave.

iii) Only concave and convex functions have the property that each of their tangent lines intersect their graph exactly once.

Till now we were concerned only with the manner of bending of a graph. Let us now discuss the measure of the bending of a graph at a point also known as the curvature at that point. We measure the curvature at $(c, f(c))$ by the ratio

$$k(c) = \frac{f''(c)}{[1 + f'(c)^2]^{3/2}}$$

The radius of curvature at (c, f(c)) is denoted by $\rho(c)$ and is defined by $\rho(c) = \frac{1}{k(c)}$, if $k(c) \neq 0$.

You will note that the radius of curvature is positive if the function is convex at that point and is negative if the function is concave there.

How does one get this ratio? Oh well! Surely, you don't think this course covers the why of everything about functions?

7.5.2 Points of Inflection

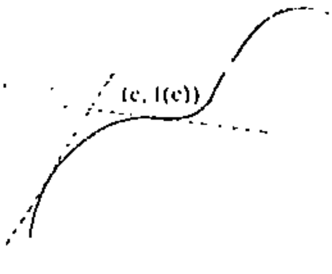


Fig. 21

Suppose $f''(c) = 0$ for some $c \in]a, b[$, and f'' changes sign in passing through c. That is, f'' has one sign on the left of c, in $]c - \delta, c[$ say, and the other on the right of c, in $]c, c + \delta[$ say. Then on one side of (c, f(c)) the tangents are above the graph and on the other side of (c, f(c)) these are below the graph. This means that the tangent at (c, f(c)) must be crossing the graph (Also see Fig. 21).

Such points (c, f(c)) are known as the **points of inflection** of (the graph of) f.

Note that we did not really use the fact that $f''(c)$ is zero. In fact $f''(c) = 0$ is neither a necessary nor a sufficient condition as you will find out from Examples 21 and 22. A function may have a point of inflection at x = c without $f''(c)$ existing at all. For the tangent to exist, the existence of $f'(c)$ is enough. However, note that if $f''(c)$ exists in $]c - \delta, c + \delta[$ and changes sign in passing through c, then $f''(c)$ must be zero.

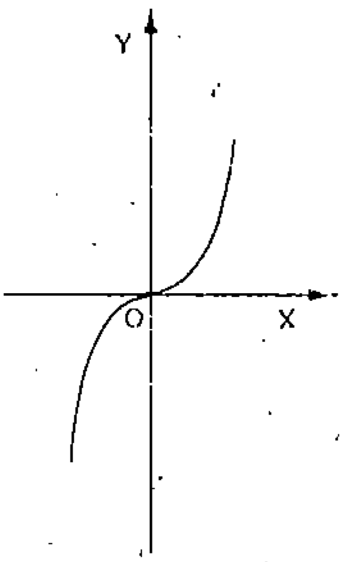


Fig. 22

Example 21 Consider the function $f(x) = x|x|$, for all $x \in \mathbf{R}$ shown in Fig. 22. This function can be rewritten as

$$f(x) = \begin{cases} x^2 \text{ if } x \geq 0 \\ -x^2 \text{ if } x < 0 \end{cases}$$

Hence

$$f'(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ -2x \text{ if } x < 0 \end{cases}$$

But $f''(0)$ does not exist and ,

$$f''(x) = \begin{cases} 2 \text{ if } x > 0 \\ -2 \text{ if } x < 0 \end{cases}$$

Thus f'' changes sign at 0 and the tangent crosses the graph at 0. x = 0 is thus a point of inflection even though $f''(0) \neq 0$ (it does not exist!). Hence $f''(c) = 0$ is NOT a necessary condition for f to have a point of inflection at x = c.

Example 22 Let $f(x) = x^4$. The first derivative test shows that f has a minimum at x = 0. Thus f does not have a point of inflection at x = 0 even though $f''(0) = 0$ (also see Fig. 17(b)). This happens because $f''(x) = 12x^2 > 0$ for all $x \neq 0$, and accordingly it does not change sign in passing through 0. Thus, $f''(c) = 0$ (by itself) is not sufficient either, for f to have a point of inflection at x = c.

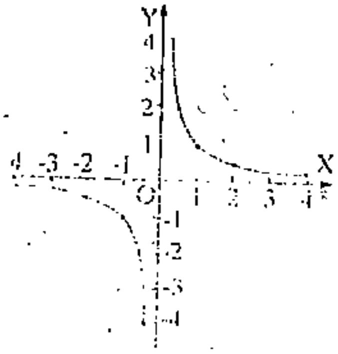


Fig. 23

Example 23 Suppose we want to find the values of x for which the graph of the function $f(x) = 1/x$, $x \in \mathbf{R} \setminus \{0\}$ is convex (i.e., concave upward) and concave (i.e. concave downward). We shall also find if the graph (Fig. 23) has any points of inflection, and the curvature at x = 1.

Here, $f'(x) = -1/x^2$, $f''(x) = 2/x^3$

Clearly, i) $f''(x) > 0$ if $x > 0$

ii) $f''(x) \neq 0$ for any $x \in \mathbf{R} \setminus \{0\}$, and

iii) $f''(x) < 0$ if $x < 0$.

It follows from (i) that the graph in question is convex (i.e., concave upward) in $]0, \infty[$. From (ii) and the fact that f'' exists for all x in the domain of f, we conclude that the graph has no points of inflection. From (iii) we deduce that the graph is concave (i.e., concave downward) in $]-\infty, 0[$.

Further, curvature at $x = 1$, is

$$\frac{f''(1)}{[1 + f'(1)^2]^{3/2}} = \frac{2}{[1 + 1]^{3/2}} = \frac{1}{\sqrt{2}}.$$

Example 24 Suppose we want to examine the function f for points of inflection if

$f(x) = x^{2n+1}$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$,

$f'(x) = (2n + 1)\, x^{2n}$, $f''(x) = (2n + 1).\, 2n.\, x^{2n-1}$

If $f''(c)$ exists, then $f''(c) = 0$ is a necessary condition for $(c, f(c))$ to be a point of inflection

Therefore, $f''(x) = 0 \implies x = 0$. Since $f''(x)$ exists for all x, there can be at most one point of inflection, namely, $(0, f(0))$, or $(0,0)$. Now to the left of $x = 0$, that is, for $x < 0$, $f''(x) < 0$, and to the right of $x = 0$, $f''(x) > 0$. Hence, f'' changes sign (from negative to positive) in passing through the origin. Thus the origin is a point of inflection of $f(x) = x^{2n+1}$.

E

E 21) Examine each of the following functions for concavity, convexity and points of inflection.

a) $f(x) = x^3$

b) $f(x) = x^{1/3}$

c) $f(x) = x^4 - 2x^3 - 12x^2 + 1$

d) $f(x) = (x - 2)/(x - 3)$

e) $f(x) = \ln x,\ x > 0$

f) $f(x) = \cos x,\ 0 < x < 2\pi$

E E 22) Find the curvature at an arbitrary point of the graph of the function

a) $f(x) = x-5, x \in R$

b) $f(x) = x^2 + 9, x \in R$

c) $f(x) = \sin x, x \in R$

d) $f(x) = \sqrt{(1 - x^2)}, -1 < x < 1$

7.6 SUMMARY

In this unit we have discussed the following points.

1) A function f is said to have a maximum (resp., minimum) value at a point c of its domain if there exists a positive number δ such that for all $x \in]c - \delta, c + \delta[$, $f(x) \leq f(c)$ (resp. $f(x) \geq f(c)$). Maximum and minimum values are known as the extreme values of the function.

2) At an extreme point c of a function f, either the derivative does not exist or is zero.

3) Critical points for a function are those where either the derivative does not exist or else has the value zero. All extreme points are critical points. A critical point may fail to be an extreme point.

4) Rolle's and Lagrange's mean value theorems, and their geometrical interpretation.

5) A sufficient condition for a function f to have an extreme value at $x = c$ is that f is continuous at c and the derivative f' changes sign in passing through c. If the change is from positive to negative, c is a maximum point. In the other event, c is a minimum point. This test is known as the first derivative test.

6) Second derivative test, another sufficient condition for the existence of extreme points asserts that if $f'(c) = 0$ then $f''(c) > 0$ implies f has a minimum at $x = c$ and $f''(c) < 0$ guarantees a maximum value at c.

7) If $f''(x) > 0$ on some interval then f is convex on it. If $f''(x) < 0$ then f is concave on it.

8) If $f''(c) = 0$ or does not exist and f'' changes sign in passing through c, then f has a point of inflection at $x = c$. This means the tangent at (c, f(c)) crosses the graph of f at this point.

9) The radius of curvature $= \dfrac{[1 + f'(c)^2]^{3/2}}{f''(c)}$, $f''(c) \neq 0$.

7.7 SOLUTIONS AND ANSWERS

E 1) a) all point of R are maxima as well as minima.

b) no maxima or minima on R.

c) no maxima or minima on $]0, 1[$.

d) $0 \in R$ is a minimum. No maxima.

e) $x = 4$ is a minimum, $x = 16$ is a maximum.

E 2) f has a minimum at c $\Longrightarrow \exists\, \delta$ such that
$f(x) \geq f(c)\ \forall x \in]c - \delta, c + \delta[$

$$\Longrightarrow f(x) - f(c) \geq 0\ \forall x \in]c - \delta, c + \delta[$$

$$\Longrightarrow \frac{f(x) - f(c)}{(x - c)} \geq 0, c < x < c + \delta$$

and $\dfrac{f(x) - f(c)}{(x - c)} \leq 0, c - \delta < x < c$

$\Longrightarrow Rf'(c) \geq 0$ and $Lf'(c) \leq 0$. But since f is differentiable at c, $f'(c)$ exists and $Lf'(c) = Rf'(c) \Longrightarrow$ each is equal to zero.

$\Longrightarrow f'(c) = 0.$

E 3) a) $f'(x) = (x - 3) + (x - 5) = 2x - 8$
$f'(x) = 0 \Longrightarrow x = 4$
$\therefore x = 4$ is a critical point.

b) $x = \frac{1}{3}(-13 \pm \sqrt{154})$

c) $x = (2n + 1)\pi/2, n \in Z$

d) no critical points

e) $x = 0$ f) $x = 0$

g) All points x for which $0 \leq x \leq 1$ are critical points because the function is defined as

$$f(x) = \begin{cases} 1 - 2x, \text{ if } x < 0 \\ 1 \text{ if } 0 \leq x \leq 1 \\ 2x - 1 \text{ if } x > 1. \end{cases}$$

$f'(x) = 0$ if $0 < x < 1$ and f is not derivable at $x = 0$ and at $x = 1$.

h) $x = 1$.

E 4) a) Yes. $y_1 = 2\sin x \cos x = \sin 2x = 0$ if $x = \pi/2 \in [0, \pi]$

b) Yes. $f'(x) = 2x = 0$ if $x = 0 \in [-2, 2]$

c) No. $f'(x) = 3x^2 + 1 \neq 0$ for $x \in [0, 1]$. Rolle's theorem does not hold as $f(0) \neq f(1)$

d) Yes. $f'(x) = \cos x - \sin x = 0$ if $x = \pi/4 \in [0, \pi/2]$.

e) Yes. $f'(x) = \cos x + \sin x = 0$ if $x = \frac{3}{4}\pi$ or $\frac{7}{4}\pi \in [0, 2\pi]$

E 5) $c = 3/2$. Yes.

E 7)
$$ap^2 + bp + c = aq^2 + bq + c \implies ap^2 + bp = aq^2 + bq$$
$$\implies a(p^2 - q^2) + b(p - q) = 0$$
$$\implies a(p + q) + b = 0 \quad \text{(since } p \neq q)$$

$f'(x) = 2ax + b$

$$f'\left(\frac{p+q}{2}\right) = a(p + q) + b = 0$$

E 8) Suppose $f(x)$ is not $(1 - 1)$ on $[x_0, \infty[$

$\implies p, q \in [x_0, \infty[$, such that $p \neq q$ and $f(p) = f(q)$

$$f'\left(\frac{p+q}{2}\right) = 0 \text{ by E 7).}$$

$\frac{p+q}{2} = x_0$ as x_0 is the unique point with $f'(x_0) = 0$

Therefore either $p < x_0$ or $q < x_0$, since p and q both cannot be equal to x_0.
This is a contradiction as we have taken $p, q \in [x_0, \infty[$.

E 9) Suppose $p, q \in I$ s.t. $p \neq q$ and $f(p) = f(q)$
If $p < q$ we have $[p, q] \subset I$, f is differentiable on $[p, q]$ and $f(p) = f(q)$

Thus f satisfies the conditions of Rolle's theorem on $[p, q]$
$\implies f'(x_0) = 0$ for some $x_0 \in [p, q] \subset I$.
But this is a contradiction.
Therefore f is one-one

E 10) a) $f(-1) = 2 = f(1) \implies \frac{f(1) - f(-1)}{1 - (-1)} = 0$

$f'(x) = 2x = 0$ if $x = 0$

$\therefore \exists\, 0 \in [-1, 1]$ s.t $f'(0) = \frac{f(1) - f(-1)}{1 - (-1)}$

b) similar

E 11) a) $f(0) = 0 = f(2)$

$f'(x) = \pi \cos \pi x \implies f'(1/2) = 0.$

b) $f(0) = 3, f(2) = 11 \implies \frac{f(2) - f(0)}{2 - 0} = 4$

$f'(x) = 4x \implies f'(1) = 4$

$\exists\, 1 \in [0, 2]$ s.t. $f'(1) = \frac{f(2) - f(0)}{2 - 0}$

E 12 a) $f(0) = 0, f(1) = 1 \implies \frac{f(1) - f(0)}{1 - 0} = 1$

$f'(x) = 3x^2 = 1 \implies x = 1/\sqrt{3} \in [0, 1]$

$\therefore c = 1/\sqrt{3}$

b) $f(-1) = -1, f(0) = 0 \implies \frac{f(0) - f(-1)}{0 - (-1)} = 1$

$f'(x) = 3x^2 = 1 \implies x = -1/\sqrt{3} \in [-1, 0]$

$c = -1/\sqrt{3}$.

c) $\frac{f(1) - f(-1)}{1 - (-1)} = 1$

$c = 1/\sqrt{3}, -1/\sqrt{3}$ are two points in $[-1, 1]$

s.t. $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$.

E 13) $f'(c) = \frac{f(b) - f(a)}{b - a} = 1$

$$\frac{g_1(b) - g_1(a)}{b - a} = \frac{f(b) - f(a)}{b - a} = f'(c) = g_1'(c)$$

$$\frac{g_2(b) - g_2(a)}{b - a} = \frac{f(b) - f(a)}{b - a} + 1 = f'(c) + 1 = g_2'(c).$$

E 14) a) $y_1 = nx^{n-1}$. Slope of the chord from (0, 0) to $(2, 2^n)$ is

$\frac{y_2 - y_1}{x_2 - x_1} = 2^{n-1}$

$nx^{n-1} = 2^{n-1} \Rightarrow x^{n-1} = \frac{2^{n-1}}{n} \Rightarrow x = \frac{2}{n^{1/(n-1)}}$

point : $\left(\frac{2}{n^{1/(n-1)}}, \frac{2^n}{n^{n/(n-1)}}\right)$

b) Slope of the chord from (0,0) to (1, 1) is 1

$\therefore nx^{n-1} = 1 \Rightarrow = x^{n-1} = 1/n \Rightarrow x = \frac{1}{1^{n/(n-1)}}$

point : $\left(\frac{1}{n^{1/(n-1)}}, \frac{1}{n^{n/(n-1)}}\right)$

E 15) Let $x \in]c - \delta, c[$. Applying the mean value theorem to $[x,c]$, $\exists\, a \in]x, c[$, such that

$f(c) - f(x) = f'(a)(c - x) < 0$, since $f'(a) < 0$ and $c - x > 0$.
Hence $f(c) > f(x)$.
Similarly, if $x \in]c, c + \delta[$, then by the mean value theorem $\exists\, b \in]c, x[$, such that
$f(x) - f(c) = f'(b)(x - c) > 0$ since $f'(b) > 0$ and $x - c > 0$.
$\Rightarrow f(x) > f(c)$, or $f(c) < f(x)$.
$\Rightarrow$ c is a minimum.

E 16) a) Max. at $x = 1$, min. at $x = 3$. Extreme values are 0 and -28. $x = 0$ is not an extremum because there is a neighbourhood of 0 in which $f'(x)$ has the same sign on either side of 0.

b) Min. at $x = 1$, $x = 3$
Max. $x = -1$
Extreme values are -3, 285, 29

c) Min. at $x = 1$, max. at $x = -1/3$, extreme values are 0 and 32/27.

E 17) a) $x = 0$; min. point. extreme value = 0

b) There are no extreme points.

c) $x = -7/6$; min.-point; E.V $= -37/12$

d) $f'(x) = xg(x)$ where $g(x)$ is a polynomial in x^2 with all co-efficients positive.
Hence $g(x) > 0$ for all $x \neq 0$. Therefore the only extreme point of f is $x = 0$.
Clearly, $f(0) = a_0$ and $f(x) > a_0$, $x \neq 0$. Hence 0 is a min. and E.V is a_0.

e) $g(x) = \frac{1}{f(x)} = x + \frac{1}{x}$. Extreme points of g are ± 1, 1 being a min.

and -1, a max. Hence 1 is a max. and -1 is a min. for f. E.V. $= \pm 1/2$.

E 18) max.

E 19) If a and b are the sides of the inscribed rectangle,

$a^2 + b^2 = d^2 \implies b = \sqrt{d^2 - a^2}$.

Area $= A = ab = a\sqrt{d^2 - a^2}$.

$A' = 0$ if $a = d/\sqrt{2}$

$A'' < 0$ for $a = d/\sqrt{2} \implies a = d/\sqrt{2}$ is a max.

$a = d/\sqrt{2} \implies b = d/\sqrt{2} \implies$ the rectangle is a square.

E 20) Suppose the display area is a rectangle with sides a cm and b cm. Then the dimensions of the name plate are $a + 2$ cm and $b + 4$ cm.

$ab = 48 \implies b = 48/a$.

$A = (a + 2)(b + 4) = (a + 2)(48/a + 4)$

$$\frac{dA}{da} = 4 - \frac{96}{a^2} = 0 \implies a^2 = \frac{96}{4} = 24 \implies a = 2\sqrt{6}$$

$$\implies b = 4\sqrt{6}$$

$$\frac{d^2A}{da^2} = \frac{198}{a^3} > 0 \text{ if } a = 2\sqrt{6} \implies \text{This is a minimum,}$$

Dimensions of the plate: $2(1 + \sqrt{6})$, $4(1 + \sqrt{6})$

E 21) a) Convex in $]0, \infty[$; concave in $]-\infty, 0[$; point of inflection (P.I) $= (0, 0)$.

b) Convex in $]-\infty, 0[$; concave in $]0, \infty[$: P.I. : 0, 0)

c) Convex in $]-\infty, -1[\cup]2, \infty[$; concave in $]-1, 2[$, P.I.: -1, $(-1, -12)$, $(2, -47)$

d) Convex if $x > 3$; concave if $x < 3$; no P.I.

e) Concave everywhere; no P.I.

f) Convex in $]\pi/2, 3\pi/2[$; concave in $]0, \pi/2[\cup]3\pi/2, 2\pi[$; P.I.;$(\pi/2, 0)$ and $(3\pi/2, 0)$.

E 22) a) 0, b) $\dfrac{2}{(1 + 4x^2)^{3/2}}$ c) $\dfrac{-\sin x}{(1 + \cos^2 x)^{3/2}}$ d) -1

UNIT 8 GEOMETRICAL PROPERTIES OF CURVES

Structure

8.1 INTRODUCTION

We started our study of Calculus by stating two problems. One of them was the problem of finding a tangent to a curve at a given point. In Unit 3 we have seen that the solution of this problem was instrumental in the development of differential calculus. Now having studied various techniques of differentiation, we shall once again take up this problem. The study of the tangents of a curve will then lead us to normals and asymptotes of curves, which we shall study in Sec.2 and Sec.5, respectively. In the last unit we discussed some other geometric features of functions, like maxima, minima, points of inflection and curvature. You will see that all these will prove very useful when we tackle curve tracing in the next unit.

Objectives

After studying this unit you should be able to

- obtain the equations of the tangent and the normal to a given curve at a given point
- calculate the angle of intersection of two curves at a given point of intersection
- obtain the angle between the radius vector and the tangent at a point on a given curve
- define and identify a singular point, a node and a cusp
- define asymptotes and obtain their equations.

8.2 EQUATIONS OF TANGENTS AND NORMALS

In Unit 3 we have seen how a tangent can be defined precisely with the help of derivatives. We have noted that the slope of a tangent to the curve $y = f(x)$ at (x_0, y_0) is given by $f'(x_0)$, whenever it exists. In fact, we had also obtained the equations of the tangents of some simple curves. Once we know how to find the equation of a tangent, it is easy to find one for a normal too. A normal to a curve, $y = f(x)$ at (x_0, y_0) is a line which passes through (x_0, y_0) and is perpendicular to the tangent at that point. This means that the slope of this normal will be

$$-\frac{1}{f'(x_0)}, \text{ if } f'(x_0) \neq 0.$$

A line L_1 is perpendicular to a line L_2 iff $m_1 m_2 = -1$, where m_1 and m_2 are the slopes of L_1 and L_2, respectively.

Now, what happens when $f'(x_0) = 0$? $f'(x_0) = 0$ implies that the slope of the tangent at (x_0, y_0) is zero, that is, this tangent is parallel to the x-axis. In this case, the normal (which is perpendicular to the tangent) would be parallel to the y-axis. The equation of this normal, would then be $x = x_0$.

Now let us look at various curves and try to obtain the equations of their tangents and normals.

Example 1 Consider the curve $y = 2\sqrt{ax}$, $a \neq 0$, shown in Fig. 1:

$a = 0$ gives us the trivial case when the curve is the line $y = 0$, or the x-axis.

Here, $\frac{dy}{dx} = \sqrt{\frac{a}{x}} = \frac{2a}{y}$. Thus, $\frac{dy}{dx}$ exists and is non-zero for all $y \neq 0$. Now y will be zero only if x is zero. Thus, we can find the equations of tangents and normals to this curve at any point, except the origin (0, 0). We know that the slope of the tangent at any point (x_0, y_0) will be $2a/y_0$. The slope of the normal will, therefore, be $-y_0/2a$. Thus, the equation of the tangent at (x_0, y_0) is

$$y - y_0 = \frac{2a}{y_0}(x - x_0)$$

Recall that the equation of a line through (x_0, y_0) having a slope m is $y - y_0 = m(x - x_0)$.

$$\Longrightarrow yy_0 - y_0^2 = 2ax - 2ax_0$$

$$\Longrightarrow yy_0 = 2ax + y_0^2 - 2ax_0$$

$$\Longrightarrow yy_0 = 2a(x + x_0), \text{ since } y_0^2 = 4ax_0.$$

The equation of the normal at (x_0, y_0) is

$$y - y_0 = \frac{-y_0}{2a}(x - x_0)$$

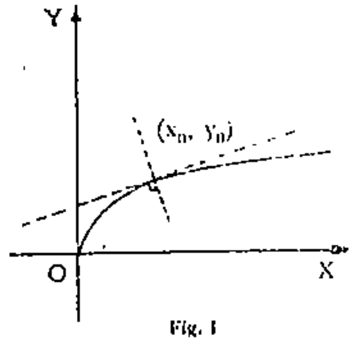


Fig. 1

Now let us see an example where the equation of the curve is given in the parametric form. In Unit 4 we have already seen what a parameter is.

Example 2 To find the equations of the tangent and the normal at the point $\theta = \pi/4$ to the curve given by $x = a\cos^3\theta$, $y = a\sin^3\theta$, (see Fig. 2), we first note that

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

Hence, the slope of the tangent at $\theta = \pi/4$ is $-\tan\pi/4 = -1$. The slope of the normal at this point thus comes out to be 1. Now, if $\theta = \pi/4$, $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = 1/\sqrt{2}$

Thus, $x = a/2\sqrt{2}$ and $y = a/2\sqrt{2}$.

The equation of the tangent at $(a/2\sqrt{2}, a/2\sqrt{2})$ is

$$y - \frac{a}{2\sqrt{2}} = -1\left(x - \frac{a}{2\sqrt{2}}\right)$$

That is, $x + y = \frac{a}{\sqrt{2}}$ or $\sqrt{2}(x + y) = a$

The equation of the normal at $(a/2\sqrt{2}, a/2\sqrt{2})$ is

$$y - \frac{a}{2\sqrt{2}} = 1\left(x - \frac{a}{2\sqrt{2}}\right)$$

or $y = x$.

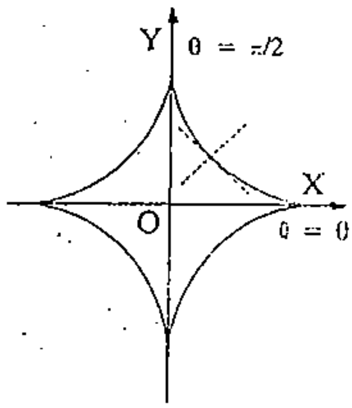


Fig. 2

Example 3 illustrates the method of finding the equations of tangents and normals when the equation of the curve is given in the implicit form.

Example 3 Let us find the equations of the tangent and the normal to the curve defined by $x^3 + y^3 - 6xy = 0$ at a point (x_0, y_0) on it.

Fig. 3 shows this curve.

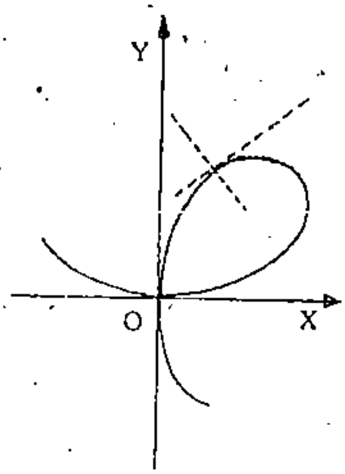


Fig. 3

In Unit 4 we have seen how to calculate the derivative when the relation between x and y is expressed implicitly. We shall follow the same procedure again. Differentiating the given equation throughout with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} - 6y - 6x \frac{dy}{dx} = 0, \text{ which means}$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Thus, the slope at the point (x_0, y_0) is $\frac{2y_0 - x_0^2}{y_0^2 - 2x_0}$.

Hence, the equation of the tangent at (x_0, y_0) is

$$y - y_0 = \frac{2y_0 - x_0^2}{y_0^2 - 2x_0}(x - x_0)$$

Simplifying, and using the relation $x_0^3 + y_0^3 = 6x_0y_0$, this reduces to

$$(2y_0 - x_0^2)x + (2x_0 - y_0^2)y + 2x_0y_0 = 0$$

Now the normal at (x_0, y_0) is a line passing through (x_0, y_0) and having slope $\frac{-(y_0^2 - 2x_0)}{2y_0 - x_0^2}$. Hence, the equation of the normal at (x_0, y_0) is

$$y - y_0 = -\frac{y_0^2 - 2x_0}{2y_0 - x_0^2}(x - x_0)$$

Simplifying, we get

$$(y_0^2 - 2x_0)x + (2y_0 - x_0^2)y + (x_0 - y_0)(2x_0 + x_0y_0 + 2y_0) = 0$$

If you have followed these examples, you should have no problem in solving the following exercises.

E 1) Find the equations of the tangent and the normal to each of the following at the specified point.

a) $y = x^2 + 2x + 1$ at $(1, 4)$

b) $x = a \cos t$, $y = b \sin t$ at the point given by $t = \pi/4$

c) $x^2 + y^2 = 25$ at $(-3, 4)$

Vertical Tangents

By now, you are quite familiar with the fact that $f'(x)$ or dy/dx may not exist at some points. At such points either the tangent does not exist, or else, is parallel to the y-axis, that is, vertical. To examine the existence of vertical tangents at (x_0, y_0), we examine $\left.\frac{dx}{dy}\right|_{y = y_0}$. If $\left.\frac{dx}{dy}\right|_{y = y_0} = 0$, then, we conclude that there is a vertical tangent at (x_0, y_0). In such cases the equation of the tangent can be written as $x = x_0$.

The normal corresponding to a vertical tangent will obviously be horizontal or parallel to the x-axis. This means we can write its equation as $y = y_0$, as its passes through (x_0, y_0).

If you take the curve in Example 2, you will find that $\frac{dy}{dx}$ does not exist when $\theta = \pi/2$.

Let us examine $\frac{dx}{dy}$ at this point. $\frac{dx}{dy} = -\cot\theta = 0$ if $\theta = \pi/2$.

This means that the curve has a vertical tangent and, consequently, a horizontal normal at this point. Now, when $\theta = \pi/2$, $x = 0$ and $y = a$. Thus the equation of the tangent at $(0, a)$ is $x = 0$ and that of the normal is $y = a$.

See if you can solve this exercise now.

E E 2) Are there any points on the following curves where the tangent is parallel to either axis? If yes, find all such points.

a) $y = x^3 - x^2 - 2x$

b) $y = \sin x$

Let us now look at another example.

Example 4 To find the equations of those tangents to the curve $y = x^3$, which are parallel to the line $12x - y - 3 = 0$, we first observe that the slope of the line $12x - y - 3 = 0$ is 12. Thus, the slope of any line parallel to this line should also be 12. Now, the slope of the tangent to the curve $y = x^3$ at any point (x, y) is $f'(x) = 3x^2$.

If we equate $f'(x)$ to 12, we will get those points on the curve where the tangent is parallel to $12x - y - 3 = 0$.

Thus, $3x^2 = 12$, or $x^2 = 4$, that is, $x = \pm 2$.

If $x = 2$, $y = x^3 = 8$. If $x = -2$, $y = x^3 = -8$.

Thus, the points in question are $(2, 8)$ and $(-2, -8)$. The equations of the tangents at these points are

$y - 8 = 12(x - 2)$ and $y + 8 = 12(x + 2)$, respectively.

The following exercises will give you some more practice in applying the concepts learned in this section.

E 3) Find the equations of the tangent and the normal to each of the following curves at the point 't':

a) $x = at^2$, $y = 2at$

b) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

E 4) Find the equation of the tangent to each of the following curves at the point (x_0, y_0).

a) $x^2 + y^2 + 4x + 6y - 1 = 0$

b) $xy = a$

E E 5) Prove that the line $2x + 3y = 1$ touches the curve $3y = e^{-2x}$ at a point whose x-coordinate is zero.

E E 6) Prove that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point $(a\sqrt{2}, b)$ is $ax + b\sqrt{2}y = (a^2 + b^2)\sqrt{2}$.

8.3 ANGLE OF INTERSECTION OF TWO CURVES

The concept of a tangent to a curve has proved very useful in describing various geometrical features of the curve. In this section we shall look at one such feature.

When two curves intersect at a point, their angle of intersection at that point can be defined with the help of their tangents there. In fact, we say that if two curves intersect at a point P, the angle of intersection of these two curves at P is an angle between the tangents to these curves at P, such that $0 \leq \theta \leq \pi/2$ (see Fig. 4).

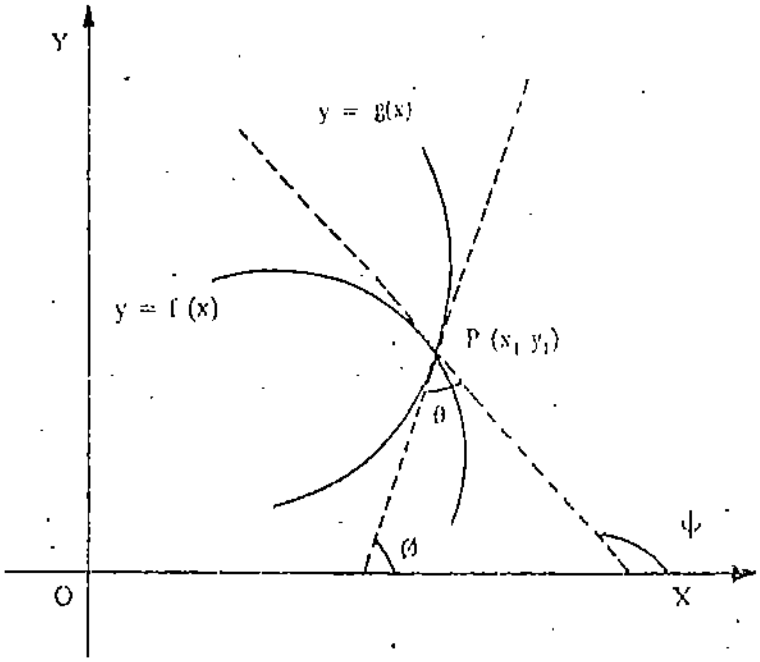


Fig. 4

We now prove a theorem which gives us the angle of intersection at a point when the equations of the two curves are known.

Theorem 1 If two curves $y = f(x)$ and $y = g(x)$ intersect at a point $P(x_1, y_1)$, then the angle θ of intersection of these curves at $P(x_1, y_1)$ is given by

$$\tan \theta = \left| \frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)} \right|$$

Proof From Fig. 4, $\tan \theta = \tan (\Psi - \Phi)$

$$= \frac{\tan \Psi - \tan \Phi}{1 + \tan \Psi \tan \Phi}$$

$$= \frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)}$$

Fig. 4 shows $\Psi - \Phi$ to be an acute angle. But if the curves f and g were as in Fig. 5, then angle $\theta = \pi - (\Psi - \Phi)$, since we take the acute angle as the angle of intersection.

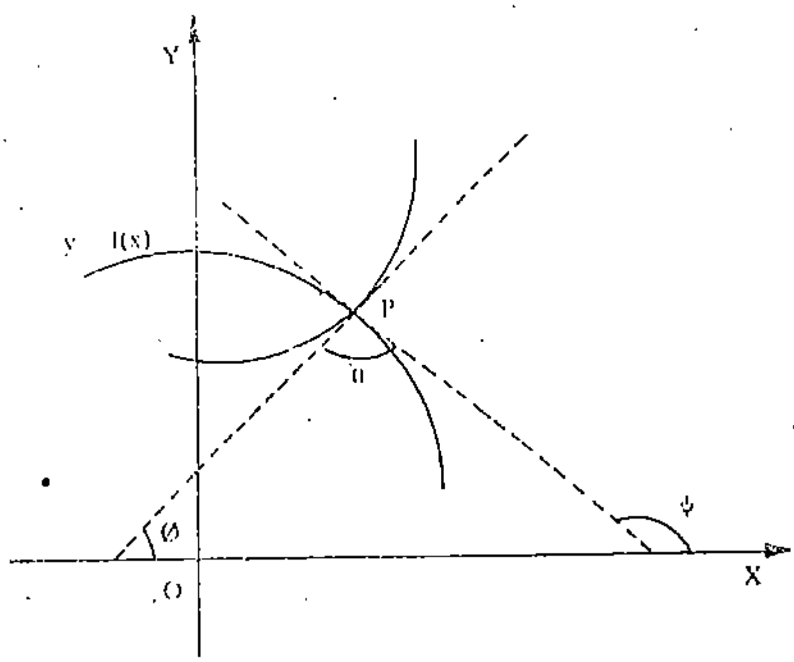


Fig. 5

In this case, $\tan\theta = \tan[\pi - (\Psi - \Phi)] = -\tan(\Psi - \Phi)$

But we are not in a position to decide whether we should take $\tan\theta$ as $\tan(\Psi - \Phi)$ or as $-\tan(\Psi - \Phi)$, unless we have drawn the curves. Since it would be tedious to first draw the curves and then decide, we think of an alternate scheme. We observe that since θ lies between 0 and $\pi/2$, $\tan\theta$ is non-negative. Thus, we take $\tan\theta$ to be $|\tan(\Psi - \Phi)|$.

Hence, $\tan\theta = \left|\dfrac{f'(x_1) - g'(x_1)}{1 + f'(x_1)\,g'(x_1)}\right|$

Having proved this theorem, we can easily deduce the following corollaries.

Corollary 1 Two curves $y = f(x)$ and $y = g(x)$ touch each other at (x_1, y_1), that is, have common tangents at (x_1, y_1), iff $\theta = 0$, that is, iff

$f'(x_1) = g'(x_1)$.

Corollary 2 Two curves cut each other at right angles, or orthogonally, at (x_1, y_1) iff $f'(x_1)\,g'(x_1) = -1$.

If you study Example 5 carefully, you will have no difficulty in solving the exercises later.

Example 5 Let us find the angle of intersection of the parabola $y^2 = 2x$ and the circle $x^2 + y^2 = 8$.

First we find the points of intersection of these curves, if there are any. The coordinates of these points will satisfy both the equation to the parabola and the equation to the circle. So substituting $y^2 = 2x$ in $x^2 + y^2 = 8$, we get

$x^2 + 2x = 8$, or $x = -4$ or 2.

It is clear from $y^2 = 2x$ that the abscissa $x\ (= y^2/2)$ of any common point must be non-negative. So we reject the value -4 of x. When $x = 2$, $y = \pm 2$. Hence the common points are P(2, 2) and Q(2, − 2). Since both curves are symmetric about the x-axis (see Fig. 6) and since P and Q are reflections of each other w.r.t. the x-axis, it is sufficient to find the angle at one point, say P; the angle at Q being equal to the angle at P. Differentiating the two equations w.r.t. to x, we get

$2y\dfrac{dy}{dx} = 2$ and $2x + 2y\dfrac{dy}{dx} = 0$

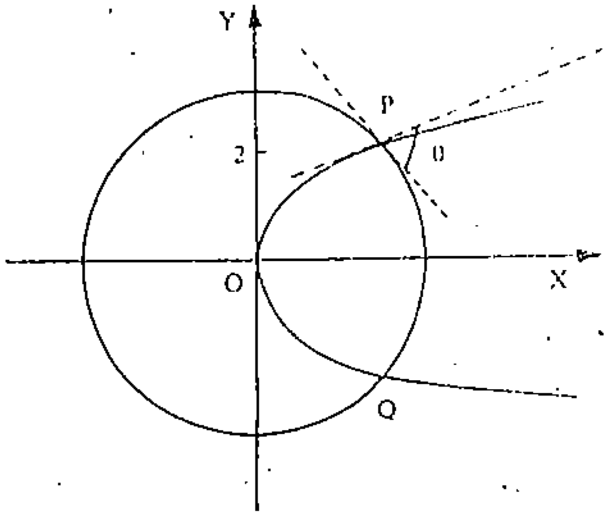


Fig. 6

Hence the values of $f'(x)$ and $g'(x)$, that is, the slopes of the tangents to the two curves at (x,y) are $1/y$ and $-x/y$. Hence the slopes of the tangents at (2, 2) to the two curves are $1/2$ and -1. Hence if θ is the required angle, then

$$\tan\theta = \left|\frac{1/2-(-1)}{1+1/2\,(-1)}\right| = 3$$

Hence, $\theta = \tan^{-1}3 = 71.56°$

You can try these exercises now.

E E 7) Find the angle of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$.

E E 8) Show that

a) the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ cut each other orthogonally (at right angles) at four points.

b) the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other (have a common tangent) at two points.

You know that given a pair of axes in a plane, the position of a point in the plane can be determined if we know its distances from the x-and y-axes. There is one more way in which we can determine its position. Suppose we are given an initial line OX in a plane (see Fig. 7(a)). Then a point P can be located if we know

i) r, its distance from O, and

ii) θ, the angle made by OP with OX.

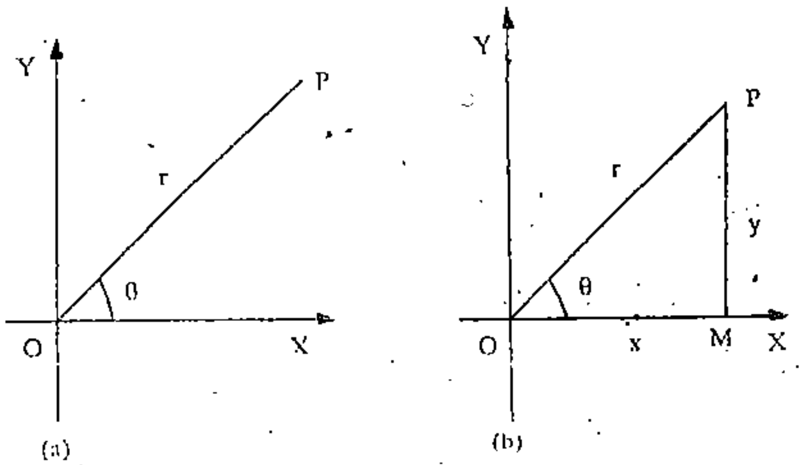


Fig. 7

r, and θ are called the polar coordinates of P. r is always taken to be non-negative, and θ takes values between 0 and 2π. From Fig. 7(b) it is clear that if x and y are the cartesian coordinates of P, then $x = r\cos\theta$ and $y = r\sin\theta$. This also gives $r^2 = x^2 + y^2$ and tan $\theta = y/x$. The equation of a curve is sometimes expressed in polar coordinates by an equation $r = f(\theta)$. For example, the equation of a circle with centre O and radius r is $r = a$. Now let us turn once again to the problem of finding the angle of intersection of two curves.

The method that we have been following till now, cannot be used if the equation of the curve is given in the polar form. In this case we follow a somewhat indirect method.

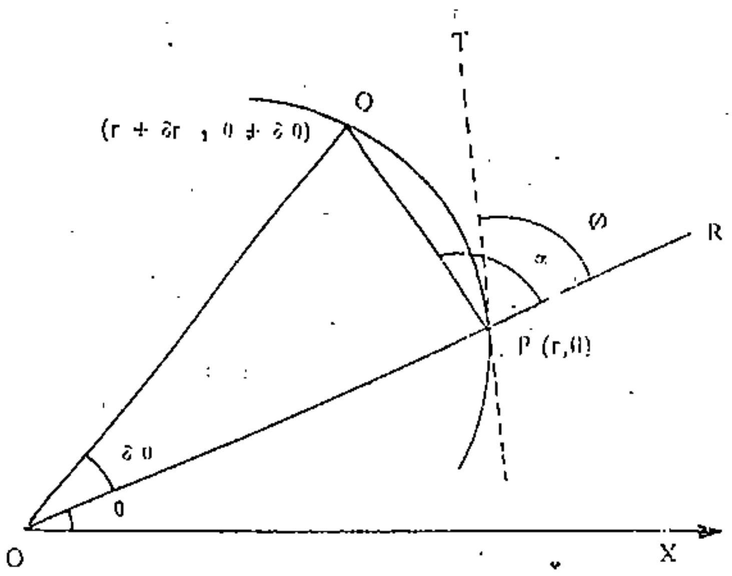


Fig. 8

Take a look at Fig. 8. It shows a curve whose equation is given in the polar form as $r = f(\theta)$. $P(r,\theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ are two points on this curve. PT is the tangent at P and OPR is the line through the origin and the point P. We shall now try to find ϕ, the angle between PT and OR.

We note here, that the tangent PT is the limiting position of the secant PQ. If we denote the angle between PQ and OR by α, then we can similarly say that ϕ is the limit of α as $Q \to P$ along the curve.

Now from ΔOPQ we have

Remember the sine rule for a Δ ABC?
$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C}$

$$\frac{OQ}{OP} = \frac{\sin \angle OPQ}{\sin \angle OQP}$$

or $$\frac{r + \delta r}{r} = \frac{\sin(\pi - \alpha)}{\sin(\alpha - \delta\theta)}$$

or $$1 + \frac{\delta r}{r} = \frac{\sin(\pi - \alpha)}{\sin(\alpha - \delta\theta)}$$

or $$\frac{\delta r}{r} = \frac{\sin\alpha - \sin(\alpha - \delta\theta)}{\sin(\alpha - \delta\theta)} \quad (\text{since } \sin(\pi - \alpha) = \sin\alpha)$$

$\sin A - \sin B$
$= 2\sin\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)$

or $$\frac{1}{r}\frac{\delta r}{\delta\theta} = \frac{2\cos(\alpha - \delta\theta/2).\sin(\delta\theta/2)}{\sin(\alpha - \delta\theta).\delta\theta}$$

$$= \frac{2\cos(\alpha - \delta\theta/2)}{\sin(\alpha - \delta\theta)} \quad \frac{\sin(\delta\theta/2)}{\delta\theta/2}$$

As $Q \to P$, $\alpha \to \phi$, $\delta\theta \to 0$ and $\delta r \to 0$. Hence as $Q \to P$, we get

Recall $\lim_{\delta\theta \to 0} \frac{\sin(\delta\theta/2)}{\delta\theta/2} = 1$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\phi}{\sin\phi} = \cot\phi$$

so that $\tan\phi = r.\frac{d\theta}{dr}$.

This formula helps us to find the angle between OP and the tangent at the point P on the curve defined by the equation $r = f(\theta)$.

We shall use this result to find the angle between two curves C_1 and C_2 which intersect at P (say). If the angles between OP and the tangents to C_1 and C_2 at P are ϕ_1 and ϕ_2, respectively, the angle of intersection of C_1 and C_2 will be $|\phi_1 - \phi_2|$ (see Fig. 9).

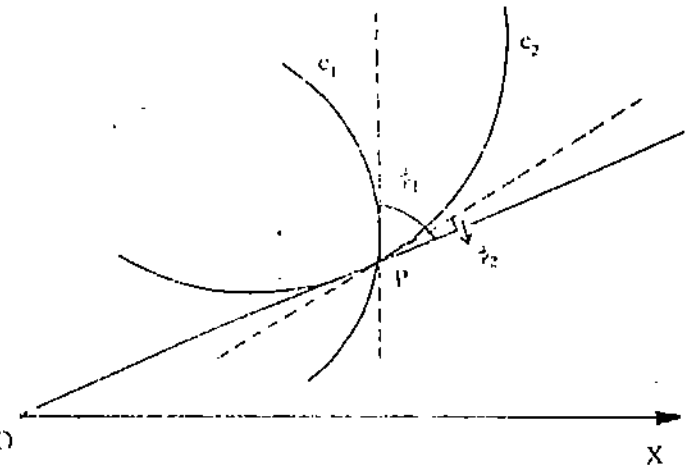


Fig. 9

This can be easily calculated as we know $\tan\phi_1$ and $\tan\phi_2$.

Thus, $\tan|\phi_1 - \phi_2| = \left|\dfrac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}\right|$

Further, if the curves intersect orthogonally, $\tan\phi_1 . \tan\phi_2 = -1$. The following examples will help clarify this discussion.

Example 6 Suppose we want to find the angle of intersection of the curves $r = a\sin 2\theta$ and $r = a\cos 2\theta$ at the point P $(a/\sqrt{2}, \pi/8)$. The coordinates of P satisfy both the equations $r = a\sin 2\theta$ and $r = a\cos 2\theta$.

If ϕ_1 is the angle between OP and the tangent to $r = a\sin 2\theta$, then

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{a\sin 2\theta}{dr/d\theta} = \frac{a\sin 2\theta}{2a\cos 2\theta} = \frac{1}{2}\tan 2\theta = \frac{1}{2}$$

Similarly, if ϕ_2 is the angle between OP and the tangent to $r = a\cos 2\theta$, then

$$\tan\phi_2 = r\frac{d\theta}{dr} = -\frac{1}{2}\cot 2\theta = -\frac{1}{2}$$

$$\text{Thus, } \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\tan\phi_2} = \frac{1/2 + 1/2}{1 - 1/4} = \frac{4}{3}$$

Thus, $\phi_1 - \phi_2 = \tan^{-1}(4/3) \approx 53.13°$, which is the required angle.

Now try to do a few exercises on your own.

E

E 9) Find the angle between the line joining a point $P(r,\theta)$ on the curve to the origin and the tangent for each of the following curves.

a) $r^2 = a^2\cos 2\theta$ b) $1/r = 1 + e\cos\theta$

c) $r^m = a^m\cos m\theta$ d) $r^m = a^m(\cos m\theta - \sin m\theta)$

E 10) Check whether the following two curves intersect orthogonally.

a) $r = ae^{\theta}$ and $re^{\theta} = b$

b) $r = a(1 + \sin\theta)$ and $r = a(1 - \sin\theta)$

8.4 SINGULAR POINTS

In this section we shall study a category of points on curves, called singular points. But to properly classify singular points we have to find the nature of tangents at these points. So let us first study on easy method of finding the tangents to a curve at the origin. This knowledge will then help us to find the tangents at any point of the curve easily.

8.4.1 Tangents at the Origin

We shall now give you a simple method of finding the tangent to a curve at the origin, when the equation of the curve is given by a polynomial equation. That is, the equation is of the form $f(x,y) = 0$ where f is a polynomial in x and y. You will agree that the constant term in this polynomial is zero since the curve passes through the origin. For such a curve the equation of the tangent at the origin can be found out by equating to zero the lowest degree terms in x and y (we shall not prove this here).

Thus, if $x^3 + 3xy + 2x + y = 0$ is the equation of a curve, the equation of the tangent at the origin is $2x + y = 0$.

Similarly, if the equation of a curve is $x^4 + x^2 - y^2 = 0$, then the equation of the tangent to this curve at (0,0) is $x^2 - y^2 = 0$, or $x^2 = y^2$, or $x = \pm y$.

Hence we get two equations $x = y$ and $x = -y$. This means that the curve $x^4+x^2-y^2 = 0$ has two tangents at the origin. We shall consider such eventualities in the next sub-section.

When the origin is shifted to (h, k) the coordinates of a point P(x, y) in the new coordinate system are given by
$x' = x - h$
$y' = y - k$

Now, consider a curve given by $g(x,y) = 0$, where $g(x, y)$ is some polynomial in x and y. Suppose we want to find the equation of the tangent to this curve at some point, say (h, k), on it. What we do is, we shift the origin to (h, k). Then with respect to this new origin, the equation of the curve will be

$g(x' + h, y' + k) = 0$. We can also express it by $G(x', y') = 0$, where
$G(x', y') = g(x' + h, y' + k)$.

Note that this change of origin does not change the shape of the curve. Now the tangent to the curve $G(x,y) = 0$ at the origin will be the tangent to the curve $g(x,y) = 0$ at the point (h, k) (see Fig. 10(a) and (b)).

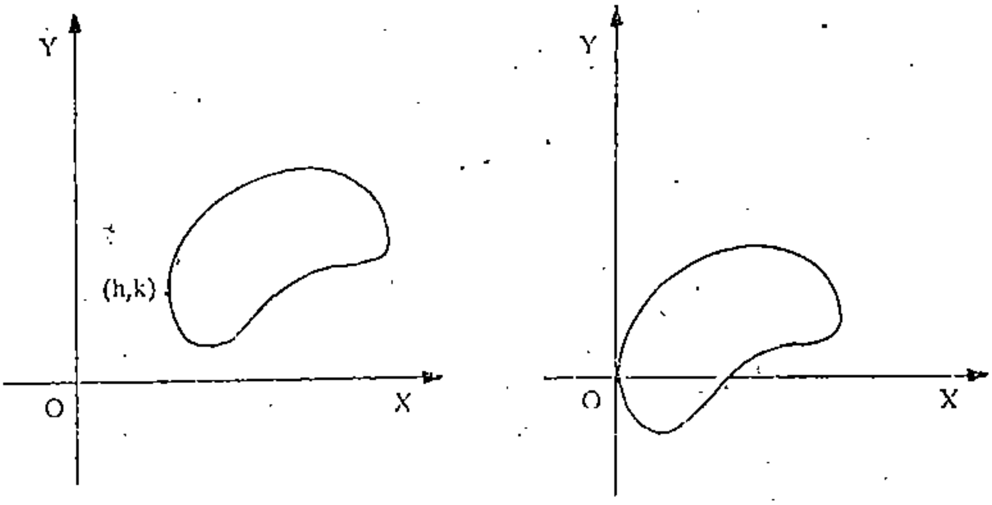


Fig. 10

The method for finding the equation of the tangent at any point of a curve will be clear to you when you read our next example.

Example 7 Consider the curve defined by the equation $ay^2 = x(x + a)^2$. Let us find the equation of the tangent to this curve at the point $(-a, 0)$.

For this, we first shift the origin to $(-a, 0)$. The equation of the curve then becomes
$ay^2 = (x - a)(x - a + a)^2$
or $ay^2 = x^2(x - a)$.

We can also write this as

$a(x^2 + y^2) = x^3$

Now, the equation of the tangent to this curve at the origin will be given by

$$a(x^2 + y^2) = 0$$
$$\implies x^2 + y^2 = 0$$
$$\implies x^2 = -y^2$$

This is impossible, since the square of any real number has to be non-negative. But we can write this as $x = \pm iy$, where $i = \sqrt{-1}$ is an imaginary number.

Thus the equations of the tangents to the given curve at the point $(-a, 0)$ are $x + a = \pm iy$ (shifting back the origin).

In such cases we say that the curve has imaginary tangents at the point $(-a, 0)$.

Now that you have seen how to find the tangents to curves given by polynomial equations, let us try and categorise the points on a given curve with the help of the tangents at those points.

Applying the procedure used in the Example 7, you should be able to solve this exercise.

E 11) Find the equations of the tangents at the origin to each of the following curves

a) $16y^2 = x^2(16 - x^2)$

b) $(y - x^2)^2 = x^4 + 3x^3$.

c) $x^3 + 6x^2y - 8y^2 = 0$.

8.4.2 Classifying Singular Points

An equation of the type $y = f(x)$ determines a unique value of y for a given value of x. This means, every straight line parallel to the y-axis meets the curve $y = f(x)$ in a unique point. However the equation of a curve is often given as $f(x, y) = 0$. If $f(x, y)$ is not a linear expression in y, then it may not be possible to write $f(x, y) = 0$ in the form $y = F(x)$ uniquely. For example, if $f(x,y) = y^3 - x^2$, then $f(x, y) = 0$ gives

$y^3 = x^2$.

This gives us two relations $y = \sqrt{a^2 - x^2}$ and $y = -\sqrt{a^2 - x^2}$ of the type $y = F(x)$.

The curve has 2 branches, as you can see from Fig. 11.

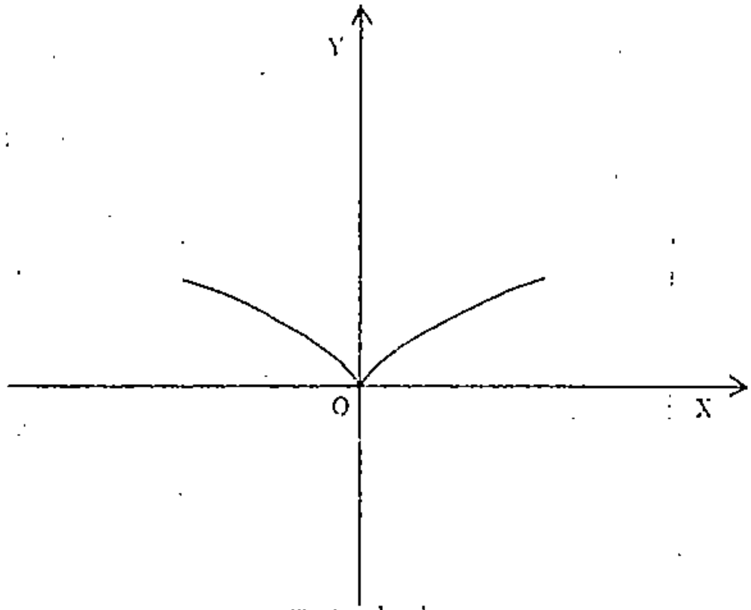


Fig. 11 : $y^3 = x^2$

The origin is common to the two branches. Put, differently we can say that two branches of the circle $x^2 + y^2 = a^2$ pass through points A and B. We have a generic name, singular points, for points like O. A precise definition is as follows.

Definition 1 If k branches of a curve pass through a point P on the curve $f(x, y) = 0$ and $k > 1$, then P is said to be a **singular point** or a **multiple point** of order k.

Singular points of order two are known as **double points**. Thus the points A and B in Fig. 11(c) are double points. Obviously, a curve will have more than one tangent at a singular point (one corresponding to each branch). Depending upon whether tangents at double points are distinct, coincident or imaginary, we shall give special names to such points.

Definition 2 A double point is known as

i) a **node** if the two tangents at that point are real and distinct,

ii) a **cusp** if the two tangents are real and coincident,

iii) a **conjugate** (or **isolated**) point if the two tangents are imaginary.

In Fig. 12 we show an example of each. For the curve $f(x, y) = 0$, the origin is a node. For the curve $g(x,y) = 0$, the points P_1, P_2, P_3 and P_4 are cusps, while the point Q on the curve $h(x, y) = 0$ is a conjugate point.

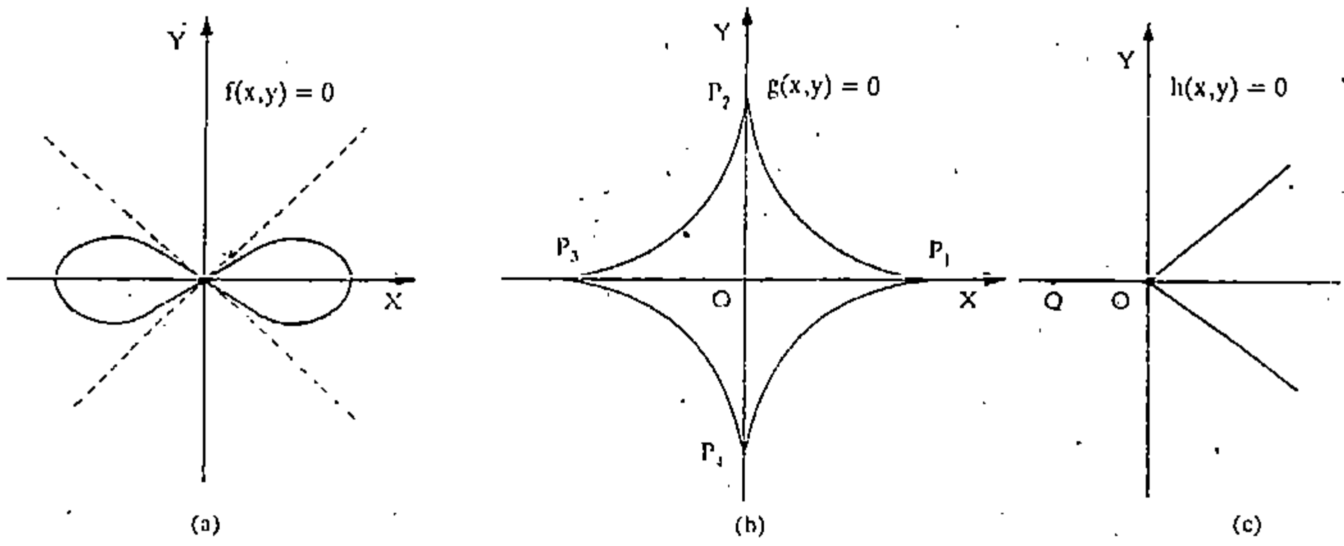


Fig. 12

In Example 7 we have seen one more example of a conjugate point. See if you can solve this exercise now.

E E 12) Show that $(-1, -2)$ is a singular point on each of the following curves.

a) $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$

b) $x^3 + xy^2 + 2(x^2 + y^2) + 4xy + 5x + 8y + 8 = 0.$

Also determine the nature (cusp, node etc.) of this singular point in both cases. (Hint : Shift the origin to $(-1, -2)$ and check the tangents at the new origin.

8.5 ASYMPTOTES

In this section we shall study another feature of curves which will prove very useful in tracing curves as you will see in the next unit. This involves taking limits as $x \rightarrow \pm\infty$ or $y \rightarrow \pm\infty$.

You have come across such limits in Unit 2. Let us define an asymptote now.

Definition 3 A straight line is said to be an asymptote to a curve, if as a point P moves to infinity along the curve, the perpendicular distance of P from the straight line tends to zero.

Example 8 Consider the rectangular hyperbola $xy = c$ shown in Fig. 13. $xy = c$ implies $y = c/x$ and this implies that as $x \rightarrow \infty$ or $-\infty$, $y \rightarrow 0$. Now $|y|$ is the distance of a point $P(x, y)$ on the hyperbola from the x-axis. So, we can say, that as $x \rightarrow \infty$ or $-\infty$, the distance of a point, P(x, y) on the hyperbola from the x-axis approaches zero. In other words, this means that the x-axis is an asymptote of the hyperbola.

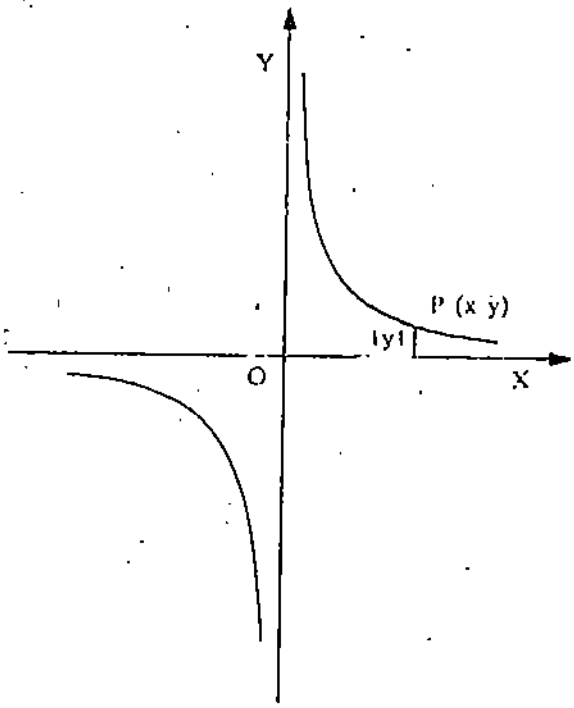


Fig. 13

Writing $xy = c$ as $x = c/y$, and repeating the arguments exactly as above, we can prove that the y-axis is also an asymptote of the hyperbola.

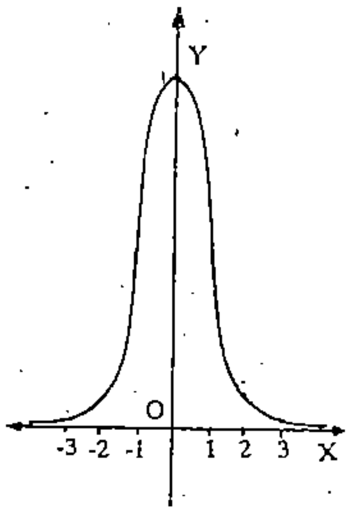


Fig. 14

Example 9 Let us prove that the x-axis is an asymptote of the curve $y = \frac{10}{1+x}$ shown in Fig. 14.

From the equation of the curve, it is quite clear that $y \to 0$ as $x \to \infty$ or $-\infty$. Again, this means that the distance of the point P(x, y) on the curve from the x-axis tends to zero as $x \to \infty$ or $-\infty$. This proves that the x-axis is an asymptote of the curve.

8.5.1 Asymptotes Parallel to the Axes

Here we shall derive tests to decide whether a given curve has asymptotes parallel to the x and y axes. For this we shall consider a curve given by $f(x, y) = 0$, where $f(x, y)$ is a polynomial in x and y.

Theorem 2 A straight line $y = c$ is an asymptote of a curve $f(x, y) = 0$ iff $y - c$ is a factor of the co-efficient of the highest power of x in $f(x, y)$.

Proof Arrange $f(x, y)$ in descending powers of x so that the equation of the curve is written as

$$g_0(y)x^n + g_1(y)x^{n-1} + \dots + g_n(y) = 0$$

$$g_0(y) + g_1(y)\frac{1}{x} + \dots + g_n(y)\frac{1}{x^n} = 0$$

P(x, y) tends to infinity means atleast one of x and y must tend to infinity.

The perpendicular distance PM of P(x, y) from the line $y = c$ is $|y - c|$. (Check this by drawing a suitable figure). Now according to Definition 3, $y = c$ is an asymptote iff PM tends to zero as P tends to infinity, that is iff $y \to c$ as P tends to infinity. If the y- coordinate y of $P \to c$ (a finite number) as P tends to infinity, then its x-coordinate x must tend to infinity. Now since P is a point on the curve, its coordinates satisfy the equation $f(x, y) = 0$

So, as P tends to infinity along the curve, we get $\lim_{P \to \infty} f(x, y) = 0$

From this we can say that $y = c$ is an asymptote iff $\lim_{\substack{x \to \infty \\ y \to c}} f(x, y) = 0$

Hence, $y = c$ is an asymptote of (1) iff

$$\lim_{\substack{x \to \infty \\ y \to c}} [g_0(y) + g_1(y)\frac{1}{x} + \dots + g_n(y)\frac{1}{x^n}] = 0$$

$\iff g_0(c) = 0,$

$\iff y - c$ is a factor of $g_0(y)$, the co-efficient of the highest power of x in $f(x, y)$.

This theorem can also be interpreted as follows.

Asymptotes parallel to the x-axis are obtained by equating to zero the real linear factors of the co-efficient of the highest power of x in the equation of the curve.

We can also state a theorem, similar to Theorem 2, giving a test to decide whether a given curve has an asymptote parallel to the y-axis or not.

Theorem 3 Asymptotes parallel to the y-axis are obtained by equating to zero the real linear factors $ax + b$ of the co-efficient of the highest power of y in the equation of the curve.

Proof : Similar to that of Theorem 2

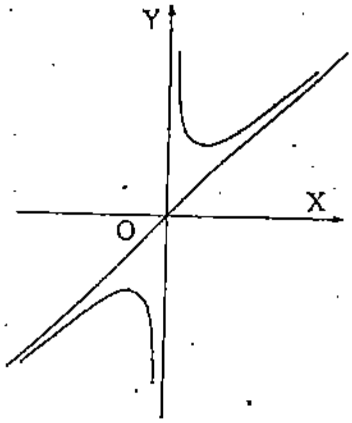


Fig. 15

Example 10 Let us find the asymptotes parallel to either axis for the curve $y = x + \frac{1}{x}$

Writing the given equation in the form $f(x, y) = 0$, we have $x^2 - xy + 1 = 0$. You can see the graph of this curve in Fig. 15. The co-efficient of the highest power of x is 1. It has no factors of the form $y - c$. Hence there are no asymptotes parallel to the x-axis. The co-efficient of the highest power of y when equated to zero gives $x = 0$. Hence there is one asymptote parallel to the y-axis and moreover, it is the y-axis itself.

See if you can do these exercises on your own.

E E 13) For each of the following curves, find asymptotes parallel to either axis, if there are any.

a) $x^2y = 2 + y$
b) $xy^2 = 16x^2 + 20y^2$
c) $(x + y)^2 = x^2 + 4$
d) $x^2y^2 = 9(x^2 + y^2)$
e) $y = \frac{1}{x^2 + 1}$
f) $y = \frac{3 - 10x}{x^2 + 10}$

8.5.2 Oblique Asymptotes

You may be wondering whether an asymptote must always be parallel to a coordinate axis. No, there are many curves having asymptotes which are not parallel to either axis. Such asymptotes are generally referred to as oblique asymptotes. We shall now learn how to find oblique asymptotes $y = mx + c$ to rational algebraic curves $f(x, y) = 0$. The problem is to determine m and c so that $y = mx + c$ may be an asymptote to $f(x, y) = 0$.

The perpendicular distance of a point $P(x_1, y_1)$ from the line $ax + by + c = 0$ is $\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$

Suppose that the line $y = mx + c$ is an oblique asymptote to the curve $f(x, y) = 0$. This means that $m = 0$. The perpendicular distance PM of a point P(x, y) on this curve from $y = mx + c$ is given by $PM = \frac{|y - mx - c|}{\sqrt{(1 + m^2)}}$ Now, since $m \neq 0$, P can be at infinity on the curve only when x (as also y), tends to ∞. Thus, as $x \to \infty$, $PM \to 0$. This means that as $x \to \infty$, $(y - mx - c) \to 0$.

or, $\lim\limits_{x \to \infty} (y - mx - c) = 0$

That is, $c = \lim\limits_{x \to \infty} (y - mx)$. .... (1)

Thus c would be known as soon as m is known. Now,

$$\lim_{x \to \infty} \left(\frac{y}{x} - m\right) = \lim_{x \to \infty} \frac{(y - mx)}{x}$$
$$= \lim_{x \to \infty} (y - mx) \lim_{x \to \infty} \left(\frac{1}{x}\right)$$
$$= c.0 = 0, \quad \text{using (1)}$$

Hence $m = \lim\limits_{x \to \infty} \left(\frac{y}{x}\right)$.

Thus, given any curve $f(x, y) = 0$, we first find $\lim\limits_{x \to \infty} \frac{y}{x} = m$ and then use this m to calculate $c = \lim\limits_{x \to \infty} (y - mx)$.

The following example will clarify this procedure.

Example 11 Let us examine the curve $x^3 - y^3 = 3xy$ for oblique asymptotes.

Suppose that the given curve has an oblique asymptote $y = mx + c$. The equation of the curve can be written as

$x^3 - y^3 - 3xy = 0$.

Dividing throughout by x^3 we get

$$1 - \frac{y^3}{x^3} - \text{[illegible]} = 0$$

Thus, $\lim\limits_{x\to\infty} [1 - \frac{y^3}{x^3} - \frac{3y}{x} \cdot \frac{1}{x}] = 0$

$$\Rightarrow 1 - \lim_{x\to\infty} (\frac{y^3}{x^3}) - 3 \lim_{x\to\infty} (\frac{y}{x}) \lim_{x\to\infty} (\frac{1}{x}) = 0$$

$$\Rightarrow 1 - \lim_{x\to\infty} (\frac{y^3}{x^3}) = 0, \text{ since } \lim_{x\to\infty} \frac{1}{x} = 0$$

$\Rightarrow m^3 = 1 \Rightarrow m = 1$, the other roots of $m^3 - 1 = 0$ being complex numbers.

Rewriting the equation of the curve as $(x - y)(x^2 + xy + y^2) = 3xy$, we have

$$c = \lim_{x\to\infty} (y - x) = \lim_{x\to\infty} \left[\frac{-3xy}{x^2 + xy + y^2}\right]$$

$$= \lim_{x\to\infty} \left[\frac{-3}{\frac{x^2}{xy} + \frac{xy}{xy} + \frac{y^2}{xy}}\right]$$

$$= \frac{-3}{1 + 1 + 1}, \text{ since } \lim_{x\to\infty} \frac{x}{y} = \lim_{x\to\infty} \left(\frac{y}{x}\right)^{-1} = 1.$$

$$= -1$$

Hence the required asymptote is $y = x - 1$.

Try to solve these exercises now.

E E 14) Find oblique asymptotes to each of the following curves.

a) $x^3 + y^3 = 3ax^2$ b) $x^4 - y^4 + xy = 0$

8.6 SUMMARY

In this unit we have covered the following points.

1) The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is
   $y - y_0 = f'(x_0)(x - x_0)$

2) The curve has a vertical tangent at (x_0, y_0) if $\frac{dx}{dy} = 0$ at this point.

3) The angle θ of intersection of two curves $y = f(x)$, $y = g(x)$ is the acute angle between the tangents at that point to the curves. It is given by the relation
   $\tan\theta = \left|\frac{f'(x) - g'(x)}{1 + f'(x)\,g'(x)}\right|$

4) $y = f(x)$ and $y = g(x)$ cut each other orthogonally at (x_0, y_0) if $f'(x_0)\,g'(x_0) = -1$.

5) The angle ϕ between the tangent and the radius vector of the curve $r = f(\theta)$ at the point θ is given by $\tan\phi = r\frac{d\theta}{dr}$.

6) The tangents at the origin to any curve (which passes through the origin) are obtained by equating to zero the lowest degree terms in the equation of the curve.

7) If k branches of a curve pass through a point P on the curve $f(x, y) = 0$ and $k > 1$, then P is said to be a singular point or a multiple point of order k. Singular points of order two are known as double points. A double point is known as a node, a cusp or a conjugate (isolated) point according as the two tangents at that point are real and distinct, real but coincident, or imaginary.

8) A straight line is said to be an asymptote to an infinite branch of a curve, if, as a point P on the curve moves to infinity along the curve, the perpendicular distance of P from the straight line tends to zero.

9) Asymptotes parallel to the coordinate axes are obtained by equating to zero the real linear factors in the co-efficients of the highest power of x and the highest power of y in the equation of the curve.

10) If $y = mx + c$ is an oblique asymptote of the curve $f(x, y) = 0$, $\lim_{x\to\infty}\frac{y}{x} = m$ and
   $\lim_{x\to\infty}(y - mx) = c$.

8.7 SOLUTIONS AND ANSWERS

E 1) a) $\left.\frac{dy}{dx}\right|_{x=1} = 4$. Equation of the tangent at (1, 4) is
   $(y - 4) = 4(x - 1)$
   Slope of the normal at $(1, 4) = -1/4$
   Equation of the normal at (1, 4) is $(y - 4) = (-1/4)(x - 1)$.

   b) Slope of the tangent $= -b/a$
   Slope of the normal $= a/b$
   at $t = \pi/4$, $x = a/\sqrt{2}$, $y = b/\sqrt{2}$. Equation of the tangent :
   $(y - b/\sqrt{2}) = -b/a\,(x - a/\sqrt{2})$

   c) Slope of the tangent $= -3/4$
   Slope of the normal $= 4/3$

Equation of the tangent : $y - 4 = (3/4)(x + 3)$
Equation of the normal : $y - 4 = (-4/3)(x + 3)$

E 2) a) Tangents are parallel to the x-axis at $x = (1 \pm \sqrt{7})/3$

b) Tangents are parallel to the x-axis at all points where $x = n\pi + \pi/2$ for some integer n. There are no tangents parallel to the y-axis.

E 3) a) Tangent : $ty = x + at^2$
Normal : $y + tx = at(2 + t^2)$

b) Tangent : $(1 + \cos t)y = \sin t(x - at)$. Equivalently,
$\sin(t/2)x - \cos(t/2)y = at \sin(t/2)$
Normal : $\sin(t/2)y + \cos(t/2)x = 2a \sin(t/2) + at \cos(t/2)$

E 4) a) $y - y_0 = -\left(\frac{x_0 + 2}{y_0 + 3}\right)(x - x_0)$

b) $y - y_0 = (-y_0/x_0)(x - x_0)$

E 5) $3y = e^{-2x} \Longrightarrow \left.\frac{dy}{dx}\right|_{x=0} = -\frac{2}{3}$. (0, 1/3) is a point on this curve. The tangent at (0, 1/3) is given by

$$y - \frac{1}{3} = -\frac{2}{3}x$$

or $2x + 3y = 1$.

E 6) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Longrightarrow \left.\frac{dy}{dx}\right|(a\sqrt{2}, b) = \frac{b\sqrt{2}}{a}$

$\Longrightarrow$ Slope of the normal $= -a/b\sqrt{2}$

$\Longrightarrow$ Equation of the normal is $y - b = \frac{-a}{b\sqrt{2}}(x - a\sqrt{2})$

E 7) $y^2 = 4x \Longrightarrow x = y^2/4 \Longrightarrow x^2 = y^4/16 = 4y$ at the point of intersection.

$\Longrightarrow y^4 - 64y = 0$
$\Longrightarrow y(y^3 - 64) = 0$
$\Longrightarrow y(y - 4)(y^2 + 4y + 16) = 0$
$\Longrightarrow y = 0, 4$ (other roots are complex)
$\Longrightarrow x = 0$ or 4.

Slope of the tangent to $y^2 = 4x$ at $(4, 4) = 1/2$
Slope of the tangent to $x^2 = 4y$ at $(4, 4) = 2$
$\Longrightarrow$ angle of intersection $= \tan^{-1}(3/4)$
The tangent at (0, 0) to $y^2 = 4x$ is vertical, and the tangent at (0, 0) to $x^2 = 4y$ is horizontal.
Hence the angle of intersection at (0, 0) is $\pi/2$.

E 8) a) The four points are $(4/\sqrt{3}, \pm \sqrt{2/3})$, $(-4/\sqrt{3}, \pm \sqrt{2/3})$

$\frac{dy}{dx}$ for $x^2 + 4y^2 = 8$ is $-x/4y$

$$\therefore \left.\frac{dy}{dx}\right|_{x=\frac{4}{\sqrt{3}},\, y=\sqrt{\frac{2}{3}}} = \frac{-1}{\sqrt{2}}$$

$\frac{dy}{dx}$ for $x^2 - 2y^2 = 4$ is $x/2y$

$$\left.\frac{dy}{dx}\right|_{x=4/\sqrt{3},\, y=\sqrt{\frac{2}{3}}} = \sqrt{2}$$

They cut orthogonally.

E 9) a) $2r = -2a^2 \sin 2\theta \frac{d\theta}{dr} \Rightarrow \frac{d\theta}{dr} = \frac{-r}{a^2 \sin 2\theta}$

$\Rightarrow$ angle $= \tan^{-1}(r \frac{d\theta}{dr}) = \tan^{-1}\left(\frac{-r^2}{a^2 \sin 2\theta}\right) = \tan^{-1}(-\cot 2\theta)$

$= \tan^{-1}(\tan(\frac{(2n+1)\pi}{2} + 2\theta)$

$\doteq (2n+1)\pi/2 + 2\theta.$

b) $\tan^{-1}(\frac{1 + e\cos\theta}{e\sin\theta})$ c) $(2n+1)\pi/2 + m\theta$

d) $m\theta - \pi/4$

E 10)a) $r = ae^{\theta} \Rightarrow 1 = ae^{\theta}.\frac{d\theta}{dr} \Rightarrow \frac{d\theta}{dr} = \frac{1}{ae^{\theta}}$

$\Rightarrow \tan\phi_1 = r\frac{d\theta}{dr} = \frac{r}{ae^{\theta}} = 1$

$re^{\theta} = b \Rightarrow r = be^{-\theta} \Rightarrow 1 = -be^{-\theta}\frac{d\theta}{dr}$

$\Rightarrow \frac{d\theta}{dr} = \frac{-1}{be^{-\theta}} \Rightarrow \tan\phi_2 = r\frac{d\theta}{dr} = \frac{-r}{be^{-\theta}} = -1$

$\Rightarrow \tan\phi_1 \tan\phi_2 = -1 \Rightarrow$ the curves cut orthogonally.

b) The curves cut orthogonally.

E 11)a) $16y^2 = 16x^2 - x^4$
The equation of the tangent is
$16y^2 - 16x^2 = 0 \Rightarrow y^2 - x^2 = 0 \Rightarrow y = \pm x.$

b) $y^2 - 2x^2y + x^4 = x^4 + 3x^3.$
The equation of the tangent is $y^2 = 0 \Rightarrow y = 0.$

c) Equation : $y^2 = 0$ or $y = 0$.

E 12)a) Change the origin to $(-1, -2)$.
Then the equation of the curve is
$(x-1)^3 + 2(x-1)^2 + 2(x-1)(y-2) - (y-2)^2 + 5(x-1) - 2(y-2) = 0.$
$\Leftrightarrow x^3 - x^2 + 2xy - y^2 = 0$
The equation of the tangents at the origin is
$x^2 - 2xy + y^2 = 0$
$\Leftrightarrow (x-y)^2 = 0 \Leftrightarrow x = y.$
There are two real and coincident tangents at this point. Hence it is a cusp.

b) After shifting the origin we get the equation :
$(x-1)^3 + (x-1)(y-2)^2 + 2[(x-1)^2 + (y-2)^2] + 4(x-1)(y-2) + 5(x-1) + 8(y-2) + 8 = 0.$
The equation of the tangents at the origin is $y^2 - x^2 = 0$, that is,
$\Leftrightarrow y^2 = x^2$ or $y = \pm x$.
There are two real and distinct tangents at this point. Hence it is a node.

E 13)a) $x^2y = 2 + y \Leftrightarrow x^2y - y - 2 = 0$
Highest power of x is 2. The coefficient of x^2 is y. Hence $y = 0$ is an asymptote.
Highest power of y is 1. The coefficient of y is $x^2 - 1 = (x-1)(x+1)$.
Hence $x = -1$ and $x = 1$ are two asymptotes.

b) No asymptotes parallel to the x-axis.
$x = 20$ is an asymptote.

c) No asymptotes parallel to the y-axis.
$y = 0$ is an asymptote.

d) $y = \pm 3$ are asymptotes.
$x = \pm 3$ are asymptotes.

e) $y = 0$ is an asymptote.

f) $y = 0$ is an asymptote.

14) a) $x^3 + y^3 = 3ax^2$

$\implies 1 + (y/x)^3 = 3a/x$

$\implies 1 + \lim_{x\to\infty} (\frac{y}{x})^3 = \lim_{x\to\infty} \frac{3a}{x}$

$\implies 1 + m^3 = 0 \implies m^3 = -1 \implies m = -1.$

$c = \lim_{x\to\infty} (y - mx) = \lim_{x\to\infty} (y + x)$

$= \lim_{x\to\infty} \frac{3ax^2}{x^2 - xy + y^2} = \lim_{x\to\infty} \frac{3a}{1 - y/x + (y/x)^2}$

$= \frac{3a}{1 + 1 + 1} = a$

Hence the equation of the asymptote is $y + x = a$.

b) $m = 1$, $c = 0$, Equation : $y = x$
$m = -1$, $c = 0$, Equation : $y + x = 0$.

UNIT 9 CURVE TRACING

Structure

9.1 INTRODUCTION

A picture is worth a thousand words. A curve which is the visual image of a functional relation gives us a whole lot of information about the relation. Of course, we can also obtain this information by analysing the equation which defines the functional relation. But studying the associated curve is often easier and quicker. In addition to this, a curve which represents a relation between two quantities also helps us to easily find the value of one quantity corresponding to a specific value of the other. In this unit we shall try to understand what is meant by the picture or the graph of a relation like $f(x, y) = 0$, and how to draw it. We shall be using many results from the earlier units here. With this unit we come to the end of Block 2, in which we have studied various geometrical features of functional relations with the help of differential calculus.

Objectives

After studying this unit you should be able to

- list the properties which can be used for tracing a curve
- trace some simple curves whose equations are given in Cartesian, parametric or polar forms.

9.2 GRAPHING A FUNCTION AND CURVE TRACING

Recall that by the graph of a function $f : D \to R$ we mean the set of points $\{(x, f(x)) : x \in D\}$. Similarly, the set of points $\{(x, y) : f(x, y) = 0\}$ is known as the graph of the functional relation $f(x, y) = 0$. Graphing a function or a functional relation means showing the points of the corresponding set in a plane. Thus, essentially curve tracing means plotting the points which satisfy a given relation. However, there are some difficulties involved in this. Let's see what these are and how to overcome them.

It is often not possible to plot all the points on a curve. The standard technique is to plot some suitable points and to get a general idea of the shape of the curve by considering tangents, asymptotes, singular points, extreme points, inflection points, concavity, monotonicity, periodicity etc. Then we draw a free hand curve as nearly satisfying the various properties as is possible.

The curves or graphs that we draw have a limitation. If the range of values of either (or both) variable is not finite, then it is not possible to draw the complete graph. In such cases the graph is not only approximate, but is also incomplete. For example, consider the simplest curve, a straight line. Suppose we want to draw the graph of $f : R \to R$ such that $f(x) = c$. We know that this is a line parallel to the x-axis. But it is not possible to draw a complete graph as this line extends infinitely on both sides. We indicate this by arrows at both ends as in Fig. 1.

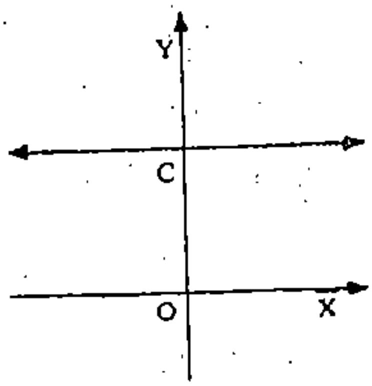


Fig. 1

In the next section we shall take up the problem of tracing of curves when the equation is given in the Cartesian form.

9.3 TRACING A CURVE : CARTESIAN EQUATION

Suppose the equation of a curve is $f(x, y) = 0$. We shall now list some steps which, when taken, will simplify our job of tracing this curve.

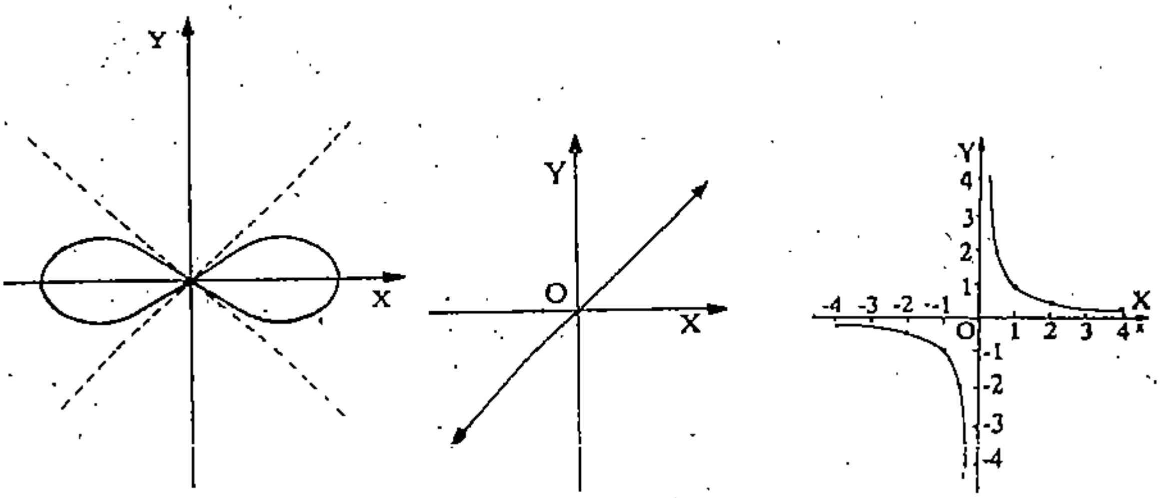
1) The first step is to determine the extent of the curve. In other words we try to find a region or regions of the plane which cannot contain any point of the curve. For example, no point on the curve $y^2 = x$, lies in the second or the third quadrant, as the x-coordinate of any point on the curve has to be non-negative. This means that our curve lies entirely in the first and the fourth quadrants.

   A point to note here is that it is easier to determine the extent of a curve if its equation can be written explicitly as $y = f(x)$ or $x = f(y)$.

A curve is symmetrical about a line if, when we fold the curve on the line, the two portions of the curve exactly coincide.

2) The second step is to find out if the curve is symmetrical about any line, or about the origin. We have already discussed symmetry of curves in Unit 1. Fig. 2, shows you some examples of symmetric curves.

A curve is symmetrical about the origin if we get the same curve after rotating it through 180°



(a) Symmetric about the x-axis. (b) Symmetric about the origin. (c) Symmetric about the line $y = x$.

Fig. 2 :

Here we give you some hints to help you determine the symmetry of a curve.

a) If all the powers of x occurring in $f(x, y) = 0$ are even, then $f(x, y) = f(-x, y)$ and the curve is symmetrical about the y-axis.

In this case we need to draw the portion of the graph on only one side of the y-axis. Then we can take its reflection in the y-axis to get the complete graph. We can similarly test the symmetry of a curve about the x-axis.

b) If $f(x, y) = 0 \iff f(-x, -y) = 0$, then the curve is symmetrical about the origin. In such cases, it is enough to draw the part of the graph above the x-axis and rotate it through 180° to get the complete graph.

c) If the equation of the curve does not change when we interchange x and y, then the curve is symmetrical about the line $y = x$. Table 1 illustrates the application of these criteria for different curves.

Table 1

Equation	Symmetry
$x^3 + y^2 + y^4 = 0$	About the x-axis (even powers of y)
$x^4 + y^3 + y^2 = 0$	About the y-axis (even powers of x)
$x^4 + x^2y^2 + y^4 = 0$	About the origin $(f(-x, -y) = 0 \iff f(x, y) = 0)$ About both axes $(f(x, y) = f(-x, y), f(x, y) = f(x, -y))$ About the line $y = x$ $(f(x, y) = f(y, x))$
$x^2 + y^4 = 10$	About both axes, (even powers of x and y) but not about $y = x$. $(f(x, y) \neq f(y, x))$

3) The next step is to determine the points where the curve intersects the axes. If we put $y = 0$ in $f(x, y) = 0$, and solve the resulting equation for x, we get the points of intersection with the x-axis. Similarly, putting $x = 0$ and solving the resulting equation for y, we can find the points of intersection with the y-axis.

4) Try to locate the points where the function is discontinuous.

5) Calculate dy/dx. This will help you in locating the portions where the curve is rising $(dy/dx > 0)$ or falling $(dy/dx < 0)$ or the points where it has a corner (dy/dx does not exist).

6) Calculate d^2y/dx^2. This will help you in locating maxima $(dy/dx = 0, d^2y/dx^2 < 0)$ and minima $(dy/dx = 0, d^2y/dx^2 > 0)$. You will also be able to determine the points of inflection $(d^2y/dx^2 = 0)$. These will give you a good idea about the shape of the curve.

7) The next step is to find the asymptotes, if there are any. They indicate the trend of the branches of the curve extending to infinity.

8) Another important step is to determine the singular points. The shape of the curve at these points is, generally, more complex, as more than one branch of the curve passes through them.

9) Finally, plot as many points as you can, around the points already plotted. Also try to draw tangents to the curve at some of these plotted points. For this you will have to calculate the derivative at these points. Now join the plotted points by a smooth curve (except at points of discontinuity). The tangents will guide you in this, as they give you the direction of the curve.

We shall now illustrate this procedure through a number of examples. You will notice, that it may not be necessary to take all the nine steps mentioned above, in each case. We begin by tracing some functions which were introduced in Unit 1.

Example 1 Consider the function $y = |x|$. Here y can take only positive values. Thus, the graph lies above the x-axis. Further, the function $y = |x|$ is symmetric about the y-axis. On the right of the y-axis, $x > 0$ and so $|x| = x$. Thus the graph reduces to that of $y = x$ and you know that this is a straight line equally inclined to the axes (Fig. 3(a) below).

The curve meets the y-axis only at the origin. Taking its reflection in the y-axis, we get the complete graph as shown in Fig. 3(b). We have drawn arrows at the end of the line segment to indicate that the graph extends indefinitely.

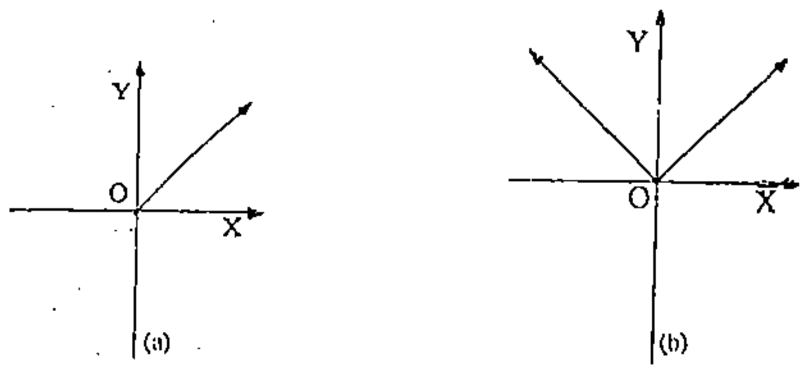


Fig. 3 : (a) Graph to the right of the y-axis. (b) Complete graph.

Example 2 The greatest integer function $y = [x]$ is discontinuous at every integer point. Hence there is a break in the graph at every integer point n. In every interval $[n, n + 1[$ its value is constant, namely n. Hence the graph is as shown in Fig. 4. Note that a hollow circle around a point indicates that the point is not included in the graph.

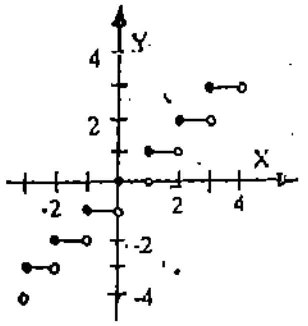


Fig. : 4 Graph of $f(x) = [x]$.

Example 3 Consider the curve $y = x^3$. Now (x, y) lies on the curve $\Longleftrightarrow y = x^3 \Longleftrightarrow -y = (-x)^3 \Longleftrightarrow (-x, -y)$ is on the curve. This means that the curve is symmetric about the origin. Thus, it is sufficient to draw the graph above the x-axis and join to it the portion obtained by rotating it through 180°.

Above the x-axis, y is positive. Hence $x = \sqrt[3]{y}$ must be positive. Thus, there is no portion of the graph in the second quadrant. The curve meets the axes of coordinates only at the origin and the tangent there, is the x-axis.

$\frac{dy}{dx} = 3x^2$ which is always non-negative. This means that as x increases, so does y. Thus the graph keeps on rising.

$\frac{dy}{dx} = 0$ at $(0, 0)$ and $\frac{d^2y}{dx^2} = 6x$ is 0 at $(0, 0)$.

$$\frac{d^2y}{dx^2} = 6x \begin{cases} > 0 \text{ for } x > 0 \\ < 0 \text{ for } x < 0 \end{cases}$$

This implies that there are no extreme points, and that $(0, 0)$ is a point of inflection. The graph has no asymptotes parallel to the axes. Further $\lim_{x \to \infty} \frac{y}{x} = \lim_{x \to \infty} x^2$ and obviously, this does not exist. This means that the curve does not have any oblique asymptotes. You can also verify that it has no singular points. The graph is shown in Fig. 5.

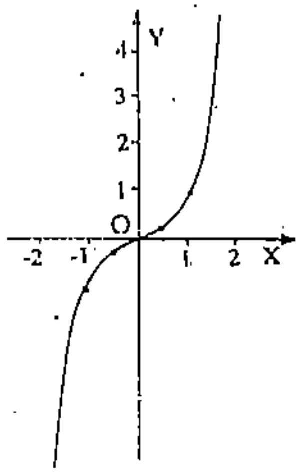


Fig. 5 : Graph of $y = x^3$

Example 4 Consider $y = \frac{1}{x^2}$. The y-coordinates of any point on the curve cannot be negative. So the curve must be above the x-axis. The curve is also symmetric about the y-axis. Hence we shall draw the graph to the right of the y-axis first.

The curve does not intersect the axes of coordinates at all.

$\frac{dy}{dx} = -\frac{2}{x^3}$ and $\frac{d^2y}{dx^2} = \frac{6}{x^4}$. Since $\frac{dy}{dx} < 0$ for all $x > 0$, the function is non-increasing in $]0, \infty[$, that is, the graph keeps on falling as x increases. Further, since $\frac{dy}{dx}$ is non-zero for all x, there are no extreme points.

Similarly, since $\frac{d^2y}{dx^2}$ is non-zero, there are no points of inflection. Writing the equation of the curve as $x^2y = 1$, we see that both the axes are asymptotes of the curve.

There are no singular points. Therefore, the curve does not fold upon itself. The curve is shown in Fig. 6.

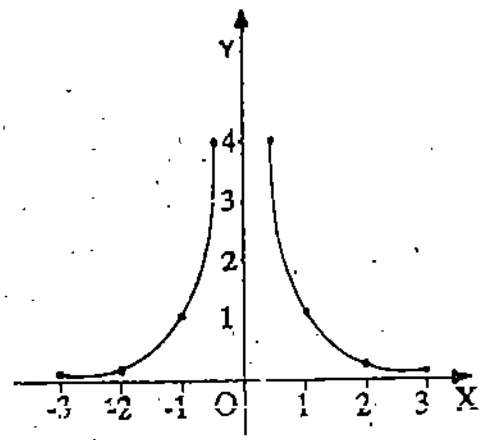


Fig. 6 : Graph of $y = 1/x^2$

Example 5 Let us try to trace the curve given by the equation $xy = 1$

Here we can see that either x and y both will be positive or both will be negative. This means that the curve lies in the first and the third quadrants.

Further, it is symmetric about the origin and hence, it is sufficient to trace it in the first quadrant and rotate this through 180° to get the portion of the curve in the third quadrant.

(1, 1) is a point on the curve and $x = 1/y$ means that as x increases in the first quadrant, y decreases.

Now the distance of any point (x, y) on the curve from the x-axis $= |y| = y = \frac{1}{x} \to 0$ as $x \to \infty$. This means that the x-axis is an asymptote. Arguing on the same lines we see that the y-axis is also an asymptote.

$\frac{dy}{dx} = \frac{-1}{x^2} \neq 0$ for any x. That is, there are no extrema.

At the point (1, 1) we have, $\frac{dy}{dx} = -1$, which implies that the tangent at (1, 1) makes an angle of 135° with the x-axis. Considering all these points we can trace the curve in the first quadrant (see Fig. 7(a)). Fig. 7(b) gives the complete curve.

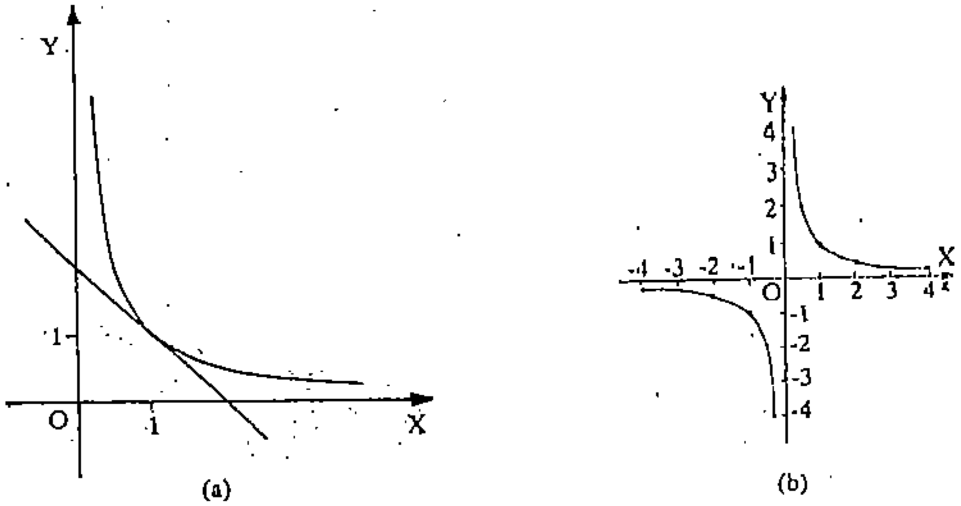


Fig. 7 (a) Graph of xy = 1 in the first Quadrant (b) complete graph

The curve traced in Example 5 is a **hyperbola**. If we cut a double cone by a plane as in Fig. 8(a), we get a hyperbola. It is a section of a cone. For this reason, it is also called a **conic section**. Figs. 8(b), (c) (d) and (e) show some other conic sections. You are already familiar with the circle in Fig. 8(d) and the pair of intersecting lines in Fig. 8(e). The curve in Fig. 8(b) is called a **parabola** and that in Fig. 8(c) is called an **ellipse**.

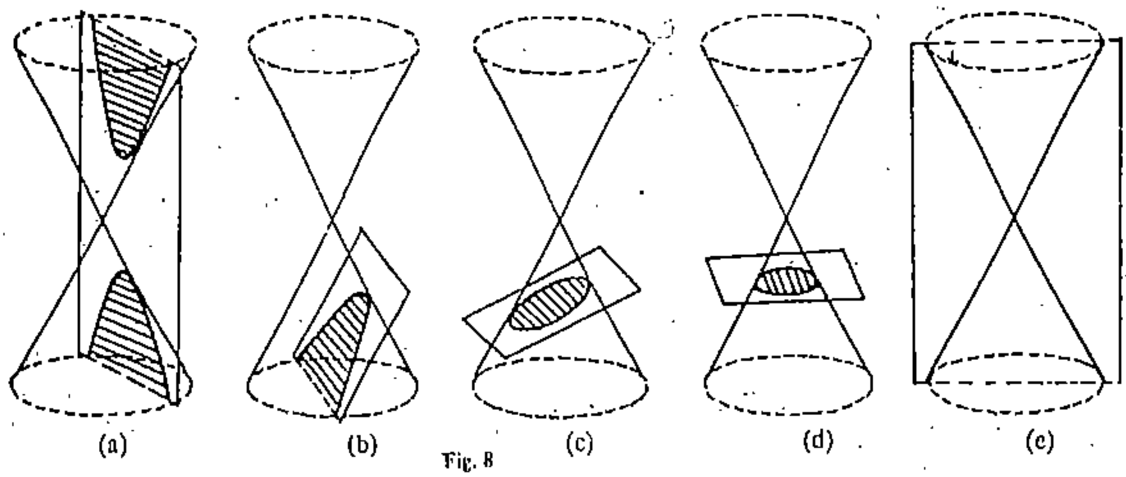


Fig. 8

The earliest mention of these curves is found in the works of a Greek mathematician Menaechmas (fourth century B.C). Later Apollonius (third century B.C.) studied them extensively and gave them their current names.

In the seventeenth century René Descartes discovered that the conic sections can be characterised as curves which are governed by a second degree equation in two variables. Blaise Pascal (1623-1662) presented them as projections of a circle. (Why don't you try this? Throw the light of a torch on a wall at different angles and watch the different conic sections on the wall). Galileo (1564-1642) showed that the path of a projectile thrown obliquely (Fig. 9) is a parabola. Paraboloid curves are also used in arches and suspension bridges (Fig. 10). Paraboloid surfaces are used in telescopes, search lights, solar heaters and radar receivers.

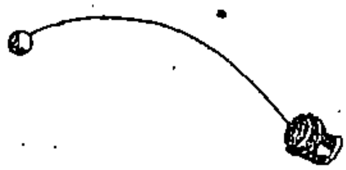


Fig. 9

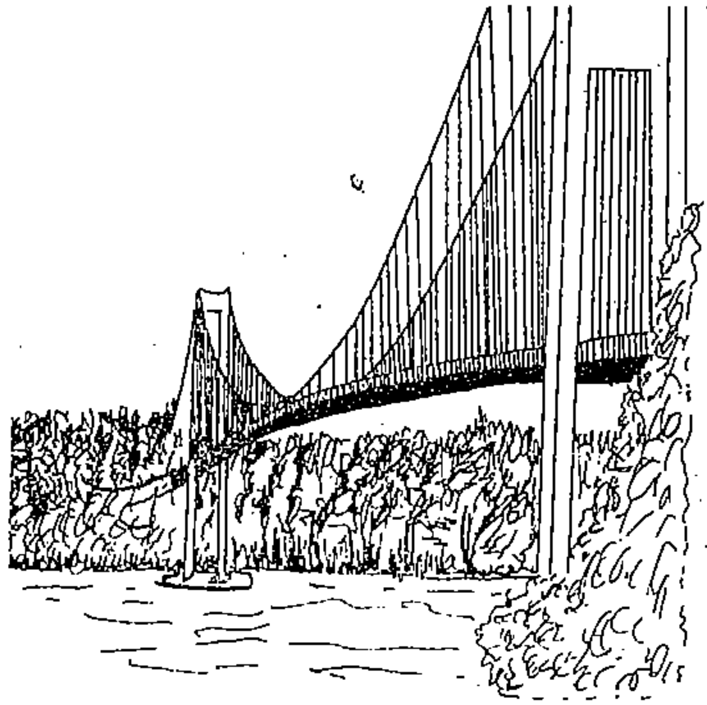


Fig. 10

In the seventeenth century Johannes Kepler discovered that planets move in elliptical orbits around the sun. Halley's comet is also known to move along a very elongated ellipse. A comet or meteorite coming into the solar system from a great distance moves in a hyperbolic path. Hyperbolas are also used in sound ranging and navigation systems.

Let's look at the next example now.

Example 6 Consider the curve $y = x^3 + x^2$.

There is no symmetry and the curve meets the axes at (0, 0) and (− 1, 0).

$\frac{dy}{dx} = 3x^2 + 2x$. The x-axis is the tangent at the origin as $\frac{dy}{dx} = 0$, at $x = 0$. Since $\frac{dy}{dx} = 1$ when $x = -1$, tangent at (− 1, 0) makes an angle of 45° with the x-axis (Fig. 11(a)).

Further $\frac{d^2y}{dx^2} = 6x + 2$. This means (0, 0) is a minimum point as $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at $x = 0$. (− 2/3, 4/27) is a maximum point as $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at $x = -2/3$. Thus in Fig. 11(b), O is a valley and P is a peak.

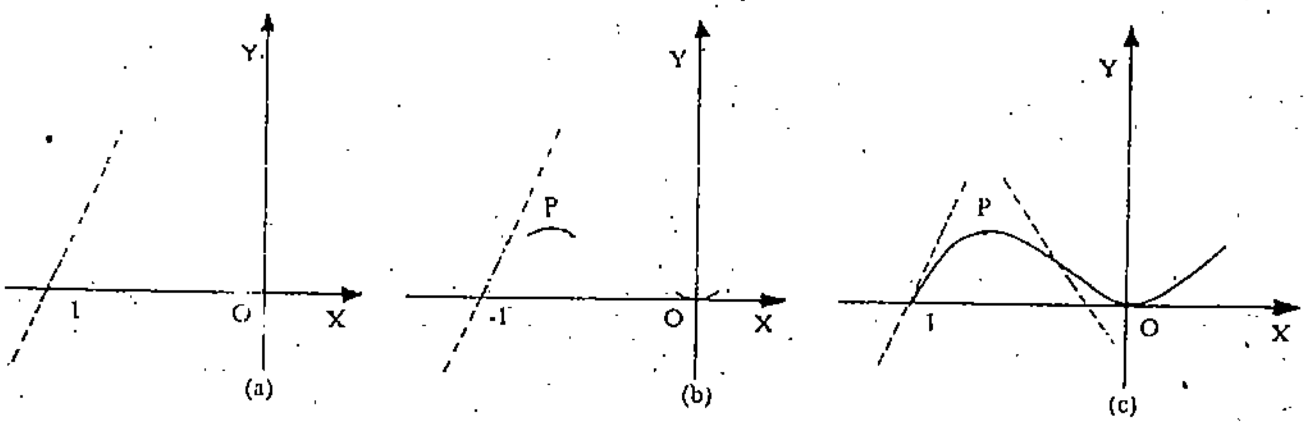


Fig. 11

$\frac{d^2y}{dx^2} = 0$ at $x = -\frac{1}{3}$ and changes sign from negative to positive as x passes through $-1/3$.

Hence $(-1/3, 2/27)$ is a point of inflection.

$\frac{dy}{dx} = x(3x + 2)$. Hence $\frac{dy}{dx} > 0$ when $x < \frac{-2}{3}$ or $x > 0$.

If $-\frac{2}{3} < x < 0$, then $\frac{dy}{dx} < 0$. Thus the graph rises in $]-\infty, -2/3[$ and $]0, \infty[$, but falls in $]-2/3, 0[$.

As x tends to infinity, so does y. As $x \to -\infty$, so does y. There are no asymptotes.

Hence the graph is as shown in Fig. 11(c).

So far, all our curves were graphs of functions. We shall now trace some curves which are not the graphs of functions, but have more than one branch.

Example 7 To trace the semi cubical parabola $y^2 = x^3$, we note that x^3 is always non-negative for points on the curve. This means x is always non-negative and no portion of the curve lies on the left of the y-axis.

There is symmetry about the x-axis (even powers of y).

The curve meets the axes only at the origin.

The tangents at the origin are given by $y^2 = 0$ so that the origin is a cusp. (see Sec. 4 in Unit 8).

In the first quadrant y increases with x and $y \to \infty$ as $x \to \infty$.

There are no asymptotes, extreme points and points of inflection.

Taking reflection in the x-axis we get the complete graph as shown in Fig. 12.

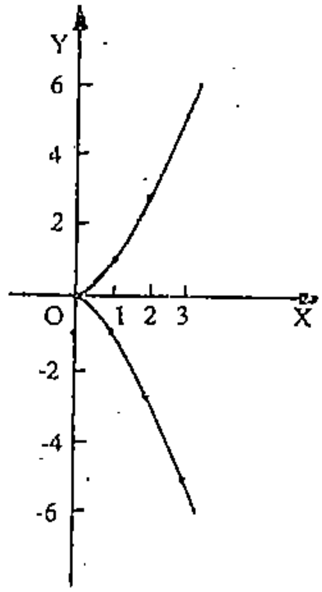


Fig. 12 : Semi cubical parabola, $y^2 = x^3$

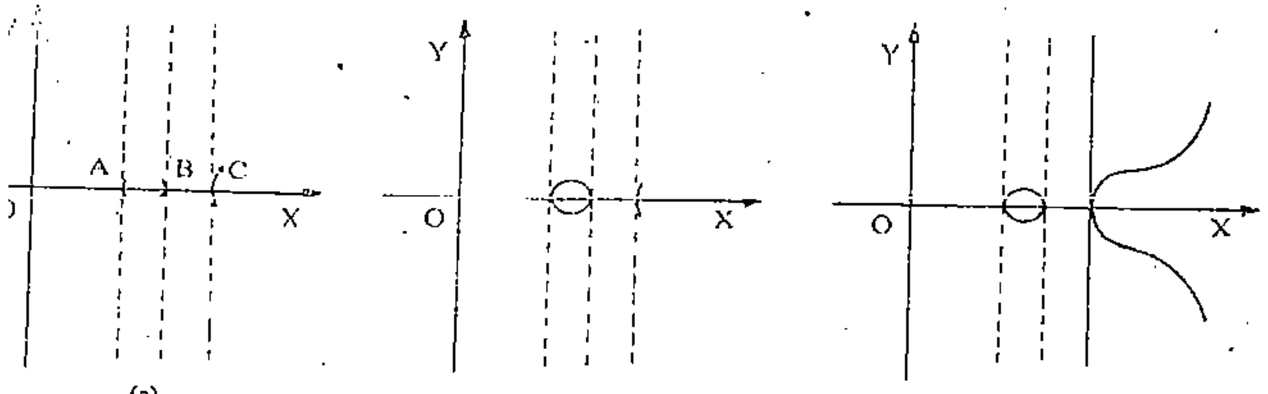
Example 8 Suppose we want to trace the curve

$y^2 = (x - 2)(x - 3)(x - 4)$.

If $x < 2$, we get a negative value for y^2 which is impossible. So, no portion of the curve lies to the left of the line $x = 2$. For the same reason, no portion of the curve lies between the lines $x = 3$ and $x = 4$.

Since y occurs with even powers alone, the curve is symmetrical about the x-axis. We may thus trace it for points above the x-axis and then get a reflection in the x-axis to complete the graph.

The curve meets the axes in points A(2, 0), B(3, 0) and C(4, 0). At each of these points, the curve has a vertical tangent (see Sec. 2 of Unit 8). Combining these facts, the shape of the curve near A, B, C must be as shown in Fig. 13(a).



(a) (b) (c)

Fig. 13

Let us take $y > 0$ (i.e., consider points of the curve above the x-axis). Then

$\frac{dy}{dx} = \frac{3x^2 - 18x + 26}{2\sqrt{(x-2)(x-3)(x-4)}}$. This is zero at $x = 3 \pm 1/\sqrt{3}$. If $\alpha = 3 + 1/\sqrt{3}$

and $\beta = 3 - 1/\sqrt{3}$ then α lies between 3 and 4, and can therefore be ignored. Also, $3x^2 - 18x + 26 = 3(x - \beta)(x - \alpha)$ and $2 < \beta < 3 < \alpha$. For $x \in]2, 3[$, $x - \alpha$ remains negative. Hence for $2 < x < \beta$, $\frac{dy}{dx} > 0$ since $(x - \alpha)$ and $(x - \beta)$ are both negative.

Similarly for $\beta < x < 3$, $\frac{dy}{dx} < 0$. Hence the graph rises in $]2, \beta[$ and falls in $]\beta, 3[$. Thus the shape of the curve is oval above the x-axis, and by symmetry about the x-axis, we can complete the graph between $x = 2$ and $x = 3$ as in Fig. 13(b).

Now let us consider the portion of the graph to the right of $x = 4$. Shifting the origin to (4, 0), the equation of the curve becomes
$y^2 = x(x+1)(x+2) = x^3 + 3x^2 + 2x$.

As x increases, so does y. As $x \to \infty$, so does y (considering points above the x-axis). When x is very small, x^3 and $3x^2$ are negligible as compared to 2x, so that near the (new) origin, the curve is approximately of the shape of $y^2 = 2x$. For large values of x, $3x^2$ and 2x are negligible as compared to x^3, so that the curve shapes like $y^2 = x^3$ for large x. Thus, at some point the curve changes its convexity.

This conclusion could also be drawn by showing the existence of a point of inflection.

There are no asymptotes or multiple points.

Considering the reflection in the x-axis, we have the complete graph as shown in Fig. 13 (c).

Example 9 Let us trace the curve $(x^2 - 1)(y^2 - 4) = 4$.

There is symmetry about both axes. We can therefore sketch the graph in the first quadrant only and then take its reflection in the y-axis to get the graph above the x-axis. The reflection of this graph in the x-axis will give the complete graph.

Notice that the origin is a point on the graph and the tangents there, are given by $4x^2 + y^2 = 0$. These being imaginary, the origin is an isolated point on the graph. The curve does not meet the axes at any other points.

For $x > 0$, $y > 0$, the equation $(x^2 - 1)(y^2 - 4) = 4$ shows that x should be greater than 1 and y should be greater than 2.

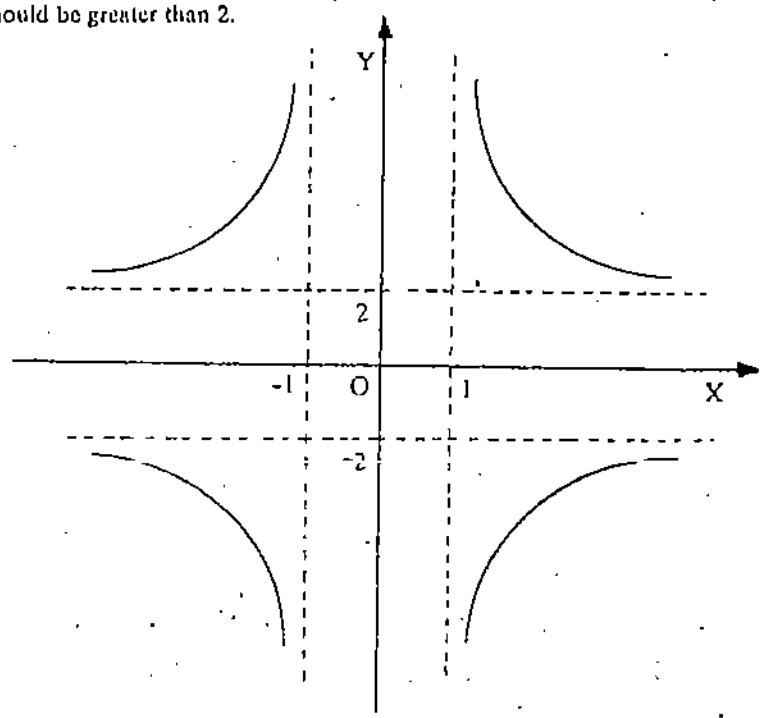


Fig. 14

Equating to zero the coefficients of the highest powers of x and y, we get $y = \pm 2$ and $x = \pm 1$ as asymptotes of the curve. Thus, the portion of the curve in the first quadrant approaches the lines $x = 1$ and $y = 2$ in the region far away from the origin.

In the first quadrant, as x increases, so does $x^2 - 1$, and since $x^2 - 1 = \frac{4}{(y^2 - 4)}$, y decreases as x increases.

There are no extreme points, singular points or points of inflection.

As $x \to \infty$, $y \to 2$ and as $y \to \infty$, $x \to 1$. Hence the graph is as shown in Fig. 14.

Example 10 To trace the curve $y^2 = (x - 1)(x - 2)^2$ we note that there is symmetry about the x-axis.

No portion of the curve lies to the left of $x = 1$.

Points of intersection with the axes are A(1, 0) and B(2, 0) and the tangent at (1, 0) is vertical. Shifting the origin to B(2, 0), the curve transforms into $y^2 = x^2(x + 1)$. The tangents at the new origin B, are given by $y^2 = x^2$. This means that B is a node, and the tangents at B are equally inclined to the axes. Let us try to build up the graph above the x-axis between $x = 1$ and $x = 2$. Differentiating the equation of the curve with respect to x, we get

$$2yy' = (x - 2)^2 + 2(x - 1)(x - 2),$$
$$= (x - 2)(3x - 4).$$

$$\text{or } y' = \frac{(x - 2)(3x - 4)}{2y}$$

when $1 < x < 2$, $(x - 2) < 0$. If y is positive, then $y' > 0$ provided $3x - 4 < 0$. Thus $y' > 0$ when $x \in]1, 4/3[$ and $y' < 0$ which $x \in]4/3, 2[$. The tangent is parallel to the x-axis when $3x - 4 = 0$, that is, when $x = 4/3$ (see Fig. 15(a)). Hence, for $1 < x < 2$, the curve shapes as in Fig. 15(b).

Now for $x > 2$. As $x \to \infty$ $y \to \infty$, $y \to \infty$ in the first quadrant. Note that when B(2, 0) is taken as the origin, the equation of the curve reduces to

$$y^2 = x^2(x + 1) = x^3 + x^2$$

This shows that when $x > 0$ and $y > 0$, the curve lies above the line $y = x$ (on which $y^2 = x^2$). Hence the final sketch (Fig. 15 (c)) shows the complete graph.

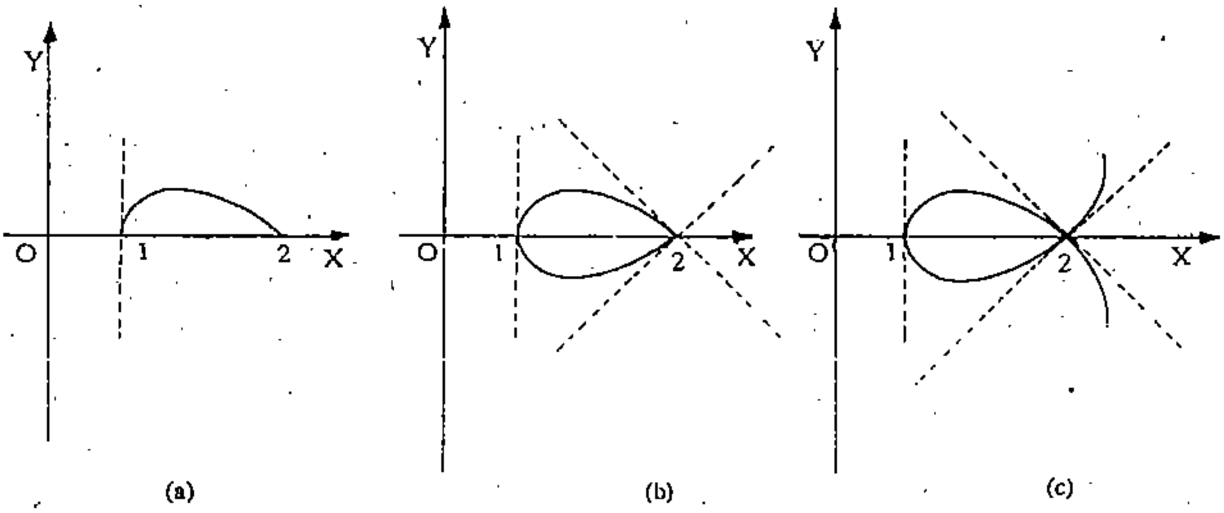


Fig. 15

If you have gone through Examples 1—10 carefully, you should be able to do the following exercise.

E 1) Trace the curves given by

a) $y = x^3$ b) $y^2 = (x - 2)^3$
c) $y(1 + x^2) = x$ d) $y^2 = x^2(1 - x^2)$

(Graph paper is provided at the end of this unit.)

9.4 TRACING A CURVE : PARAMETRIC EQUATION

Sometimes a functional relationship may be defined with the help of a parameter. In such cases we are given a pair of equations which relate x and y with the parameter. You have already come across such parametric equations in Unit 4. Now we shall see how to trace a curve whose equation is in the parametric form.

We shall illustrate the process through an example.

Example 11 Let us trace the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ as t varies from $-\pi$ to π.

$\frac{dx}{dt} = a(1 + \cos t)$, $\frac{dy}{dt} = a \sin t$, so that

$\frac{dy}{dx} = \tan(t/2)$. Since $\frac{dx}{dt} > 0$ for all $t \in]-\pi, \pi[$, x increases with t from $-a\pi$ (at $t = -\pi$) to 0 (at $t = 0$) to $a\pi$ (at $t = \pi$).

Also, $\frac{dy}{dx}$ is negative when $t \in]-\pi, 0[$ and positive when $t \in]0, \pi[$. Hence y decreases from 2a to 0 in $[-\pi, 0]$ and increases from 0 to 2a in $[0, \pi]$. Let us tabulate this data.

$t \in [-\pi, 0]$	$t \in [0, \pi]$
i) x increases from $-a$ to 0	i) x increases from 0 to a
ii) y decreases from 2a to 0	ii) y increases from 0 to 2a
iii) Hence the curve falls	iii) Hence the curve rises

Also, at the terminal points $-\pi$, 0 and π of the intervals $[-\pi, 0]$ and $[0, \pi]$, we have the following.

t	(x, y)	$\frac{dy}{dx}$	$\frac{dx}{dy}$	Tangent
$-\pi$	$(-a\pi, 2a)$	not defined	0	vertical
0	(0, 0)	0	not defined	horizontal
π	$(a\pi, 2a)$	not defined	0	vertical

On the basis of the data tabulated above, the graph is drawn in Fig. 16.

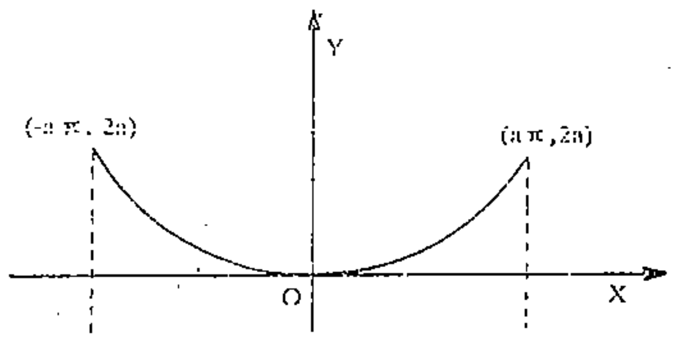


Fig. 16

Remark 1 If t is increased by 2π, x is increased by $2\pi a$ and y does not change. Thus the complete graph can be obtained in intervals ..... $[-5\pi, -3\pi]$, $[-3\pi, -\pi]$, $[\pi, 3\pi]$, $[3\pi, 5\pi]$ ..... by mere translation through a proper distance.

The cycloid is known as the Helen of geometry because it was the cause of many disputes among mathematicians. It has many interesting properties. We shall describe just one of them here. Consider this question : What shape should be given to a trough connecting two points A and B, so that a ball rolls from A to B in the shortest possible time?

Now, we know that the shortest distance between A and B would be along the line AB (Fig. 17). But since we are interested in the shortest time rather than distance, we must also consider the fact that the ball will roll quicker, if the trough is steeper at A. The Swiss mathematicians Jakob and Johann Bernoulli proved by exact calculations that the trough should be made in the form of an arc of a cycloid. Because of this, a cycloid is also called the curve of the quickest descent.

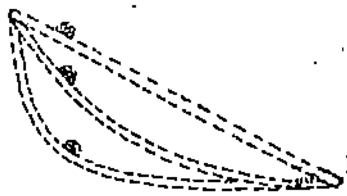


Fig. 17

The cycloid is used in clocks and in teeth for gear wheels. It can be obtained as the locus of a fixed point on a circle as the circle rolls along a straight line.

See if you can do this exercise now.

E 2) Trace the following curves on the graph paper given at the end of this unit.

a) $x = a(t + \sin t)$, $y = a(1 + \cos t)$, $-\pi \leq t \leq \pi$.

b) $x = a \sin 2t (1 + \cos 2t)$, $y = a \cos 2t (1 - \cos 2t)$, $0 \leq t \leq \pi$.

c) $x = at^2$, $y = 2at$, $0 \leq t \leq 1$.

9.5 TRACING A CURVE : POLAR EQUATION

In this section we shall consider the problem of tracing those curves, whose equations are given in the polar form. The following considerations can be useful in this connection.

Symmetry : If the equation remains unchanged when θ is replaced by $-\theta$, then the curve is symmetric with respect to the initial line.
If the equation does not change when r is replaced by $-r$, then the curve is symmetric about the pole (or the origin).

Finally if the equation does not change when θ is replaced by $\pi - \theta$, then the curve is symmetric with respect to the line $\theta = \pi/2$.

Extent : (i) Find the limits within which r must lie for the permissible values of θ. If $r < a$ ($r > a$) for some $a > 0$, then the curve lies entirely within (outside) the circle $r = a$.

(ii) If r^2 is negative for some values of θ, then the curve has no portion in the corresponding region.

Angle between the line joining a point of the curve to the origin and the tangent : At suitable points, this angle can be determined easily. It helps in knowing the shape of the curve at these points. Recall that angle ϕ is given by the relation $\tan \phi = r \frac{d\theta}{dr}$.

We shall illustrate the procedure through some examples. Study them carefully, so that you can trace some curves on your own later.

Example 12 Suppose we want to trace the cardioid $r = a(1 + \cos \theta)$. We can make the following observations.

Since $\cos \theta = \cos(-\theta)$, the curve is symmetric with respect to the initial line.

Since $-1 \leq \cos\theta \leq 1$, the curve lies inside the circle $r = 2a$.

$\frac{dr}{d\theta} = -a \sin \theta$. Hence $\frac{dr}{d\theta} < 0$ when $0 < \theta < \pi$. Thus r decreases as θ increases in the interval $]0, \pi[$. Similarly, r increases with θ in $]\pi/2, \pi[$. Some corresponding values of r and θ are tabulated below.

θ	0	$\pi/2$	π
r	2a	a	0

$r\dfrac{d\theta}{dr} = \dfrac{a(1+\cos\theta)}{-a\sin\theta} = -\cot(\theta/2) = \tan\left(\dfrac{\pi}{2} + \dfrac{\theta}{2}\right)$. This shows that the angle between the line joining a point (r, θ) on the curve to the origin and the tangent is 0 or $\pi/2$ according to $\theta = \pi$ or 0. Hence the line joining a point on the curve to the origin is orthogonal to the tangent when $\theta = 0$ and coincides with it when $\theta = \pi$.

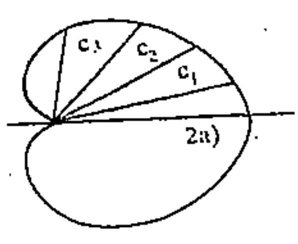


Fig. 18

Combining the above facts, we can easily draw the graph above the initial line. By reflecting this portion in the initial line we can completely draw the curve as shown in Fig 18. Notice the decreasing radii 2a, c_1, c_2, c_3 etc.

This curve is called a cardioid since it resembles a heart.

Example 13 Let us trace the equiangular spiral $r = ae^{\theta\cot\alpha}$. We proceed as follows.

When $\theta = 0$, $r = a$.

$\dfrac{dr}{d\theta} = r\cot\alpha$, which is positive, assuming $\cot\alpha > 0$. Hence as θ increases so does r.

$r\dfrac{d\theta}{dr} = \tan\alpha$. Thus, at every point, the angle between the line joining a point on the curve to the origin and the tangent is the same, namely α. Hence the name.

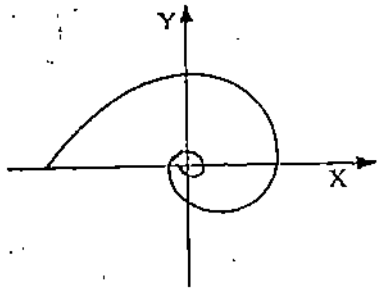


Fig. 19

Combining these facts, we get the shape of the curve as shown in Fig. 19.

The equiangular (or logarithmic) spiral $r = ae^{\theta\cot\alpha}$ is also known as the curve of pursuit. Suppose four dogs start from the four corners of a square, each pursues the dog in front with the same uniform velocity (always following the dog in front in a straight line), then each will describe an equiangular spiral. Several shells and fossils have forms which are quite close to equiangular spirals (Fig. 20). Seeds in the sunflower or blades of pine cones are also arranged in this form.

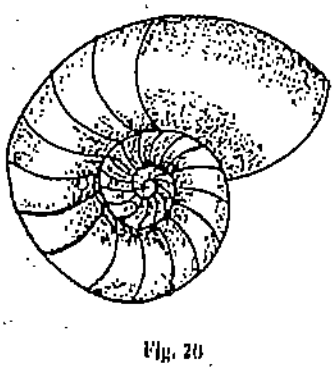


Fig. 20

This spiral was first studied by Descartes in 1638. John Bernoulli rectified this curve and was so fascinated by it that he willed that an equiangular spiral be carved on his tomb with the words 'Though changed, I rise unchanged' inscribed below it.

The spiral $r = a\theta$ is known as the Archimedean spiral. Its study was, however, initiated by Conan. Archimedes used this spiral to square the circle, that is, to find a square of area equal to that of a given circle. This spiral is widely used as a cam to produce uniform linear motion. It is also used as casings of centrifugal pumps to allow air which increases uniformly in volume with each degree of rotation of the fan blades to be conducted to the outlet without creating back-pressure.

The spiral $r\theta = a$, due to Varignon, is known as the reciprocal or hyperbolic (recall that $xy = a$ is a hyperbola) spiral. It is the path of a particle under a central force which varies as the cube of the distance.

Now let's consider one last example.

Example 14 To trace the curve $r = a\sin 3\theta$, $a > 0$, we note that there is symmetry about the line $\theta = \pi/2$, since the equation is unchanged if θ is replaced by $\pi - \theta$.

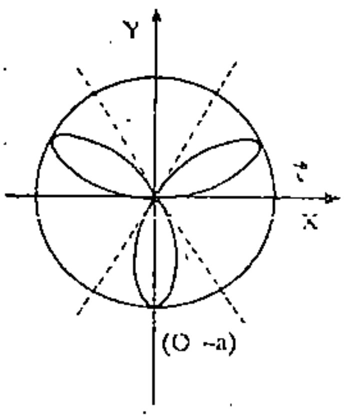


Fig. 21

The curve lies inside the circle $r = a$, because $\sin 3\theta \leq 1$. The origin lies on the curve and this is the only point where the initial line meets the curve.

$r = 0 \Longrightarrow \theta = n\pi/3$, where n is any integer. Hence the origin is a multiple point, the lines $\theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3, 2\pi$ etc. being tangents at the pole.

$\dfrac{dr}{d\theta} = 3\cos 3\theta$. Hence r increases in the intervals $]0, \pi/6[$, $]\pi/2, 5\pi/6[$, and $]7\pi/6, 3\pi/2[$, and decreases in the intervals $]\pi/6, \pi/2[$, $]5\pi/6, 7\pi/6[$ and $]3\pi/2, 5\pi/3[$. Notice that r is negative when $\theta \in]\pi/3, 2\pi/3[$ or $\theta \in]\pi, 4\pi/3[$ or $\theta \in]5\pi/3, 2\pi[$.

Hence the curve consists of three loops as shown in Fig. 21. The function is periodic and the curve traces itself as θ increases from 2π on.

Now try to trace a few curves on your own.

E 3) Trace the following curves on the graph paper provided.

a) $r = a(1 - \cos\theta), a > 0$. b) $r = 2 + 4\cos\theta$.

c) $r = a\cos 3\theta, a > 0$. d) $r = a\sin 2\theta, a > 0$

(Graph paper is provided at the end of this unit.)

9.6 SUMMARY

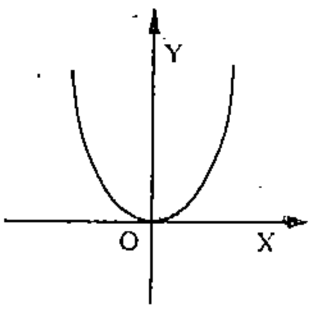
In this unit we have covered the following points.

1) Tracing a curve $y = f(x)$ or $f(x, y) = 0$ means plotting the points which satisfy this relation.
2) Criteria for symmetry and monotonicity, equations of tangents, asymptotes and points of inflection are used in curve tracing.
3) Curve tracing is illustrated by some examples when the equation of the curve is given in
   a) Cartesian form
   b) Parametric form
   c) Polar form

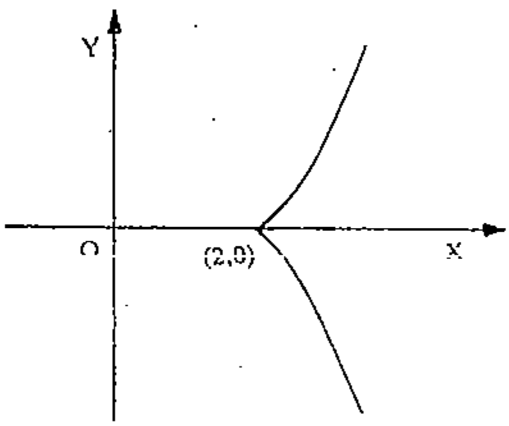
9.7 SOLUTIONS AND ANSWERS

Dotted lines represent tangents or asymptotes throughout.

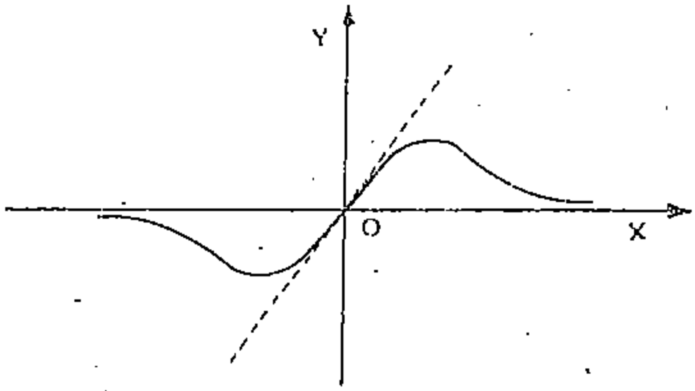
E 1) a)



b) Shifting the origin to (2, 0) we get $y^2 = x^3$ which you know how to draw.



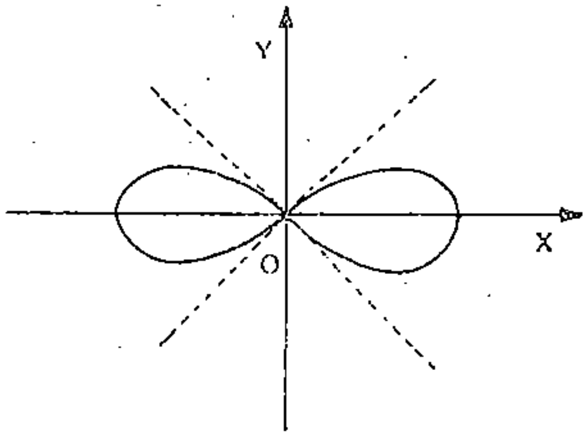
c) $y = x$ is the tangent at the origin $(0, 0)$. $(\sqrt{3}, \frac{\sqrt{3}}{4})$, $(-\sqrt{3}, -\frac{\sqrt{3}}{4})$ are point of infections, x-axis is an asymptote. Either x, y are both positive or both negative. Function rises in $]-1, 1[$ and falls elsewhere. Graph is shown alongside.



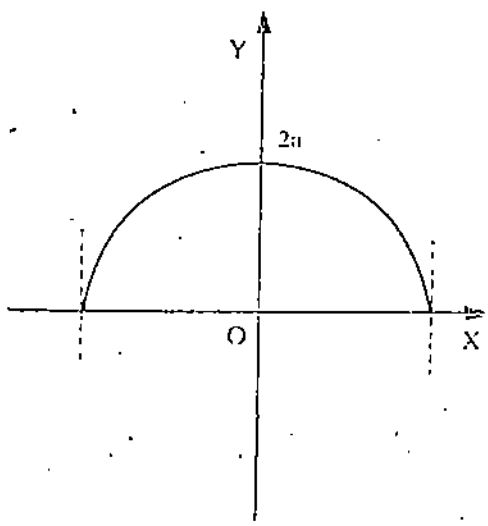
d) $\frac{y^2}{x^2} = 1 - x^2$ shows that the entire curve lies within the lines $x = \pm 1$.

Tangents at the origin are $y = \pm x$. Tangents at $x = \pm 1$ are vertical.

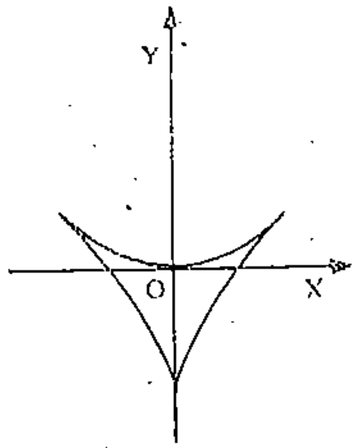
Maxima at $(\pm 1/\sqrt{2}, 1/4)$, symmetry about both axes.



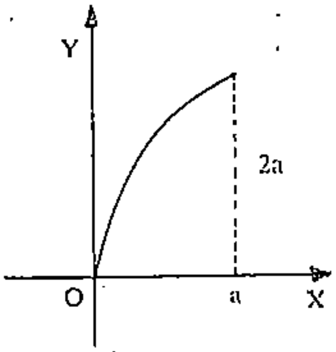
E 2) a)



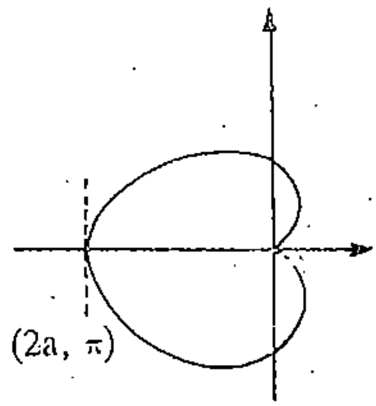
b)



c)



E 3) a)



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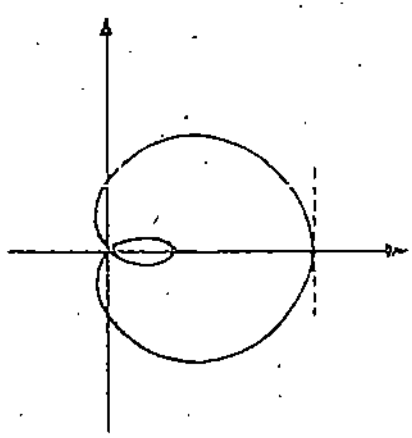

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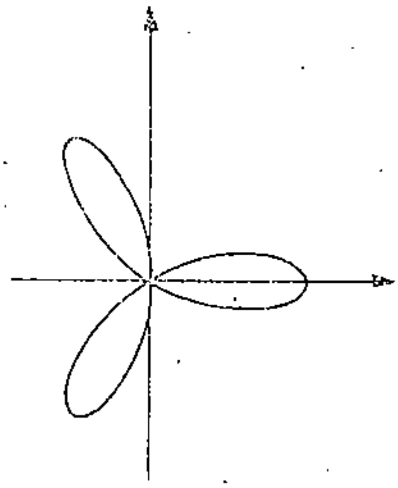
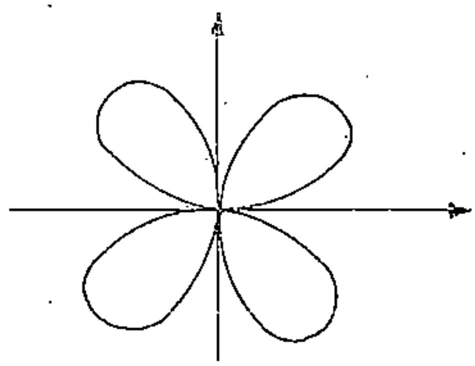
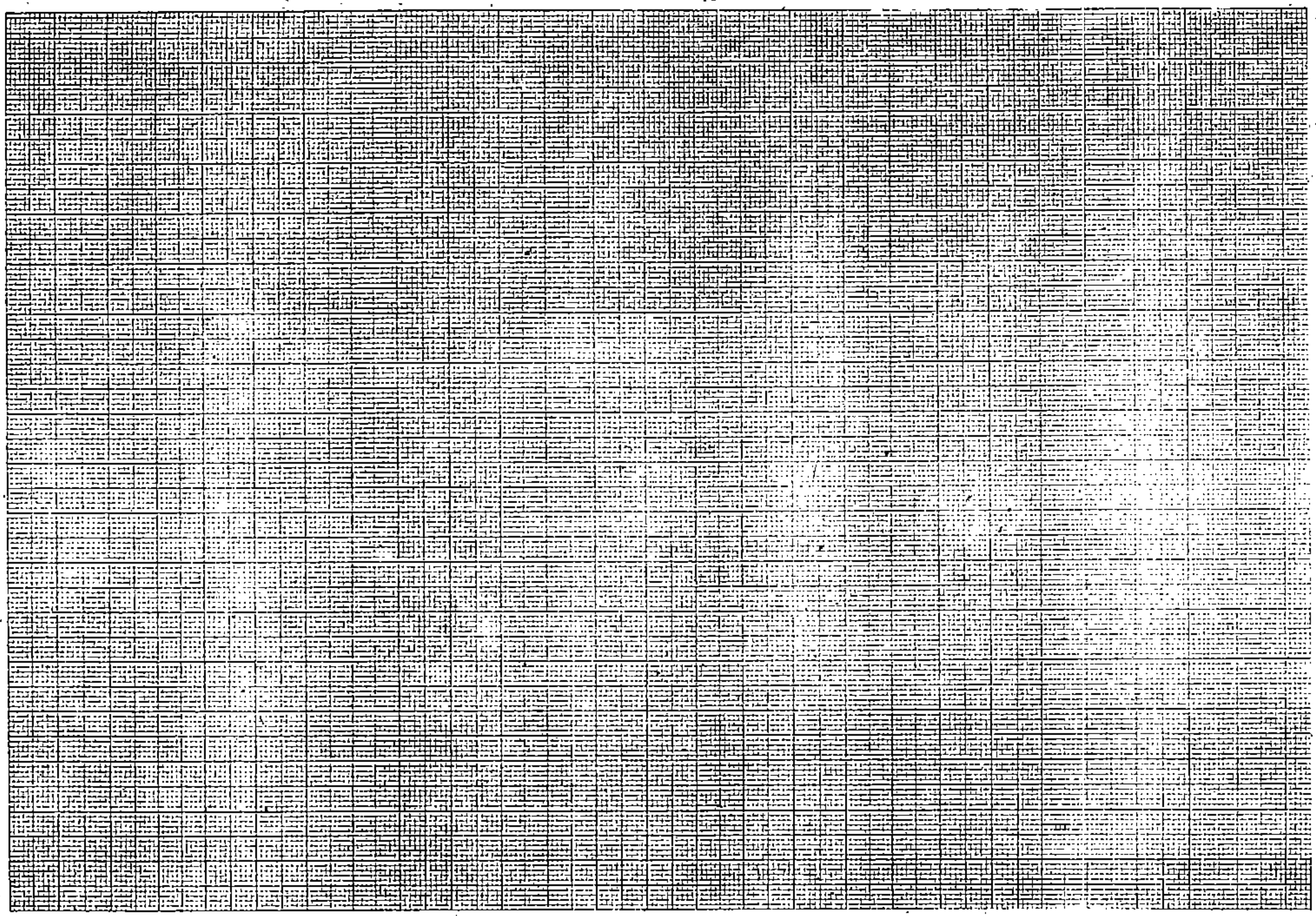
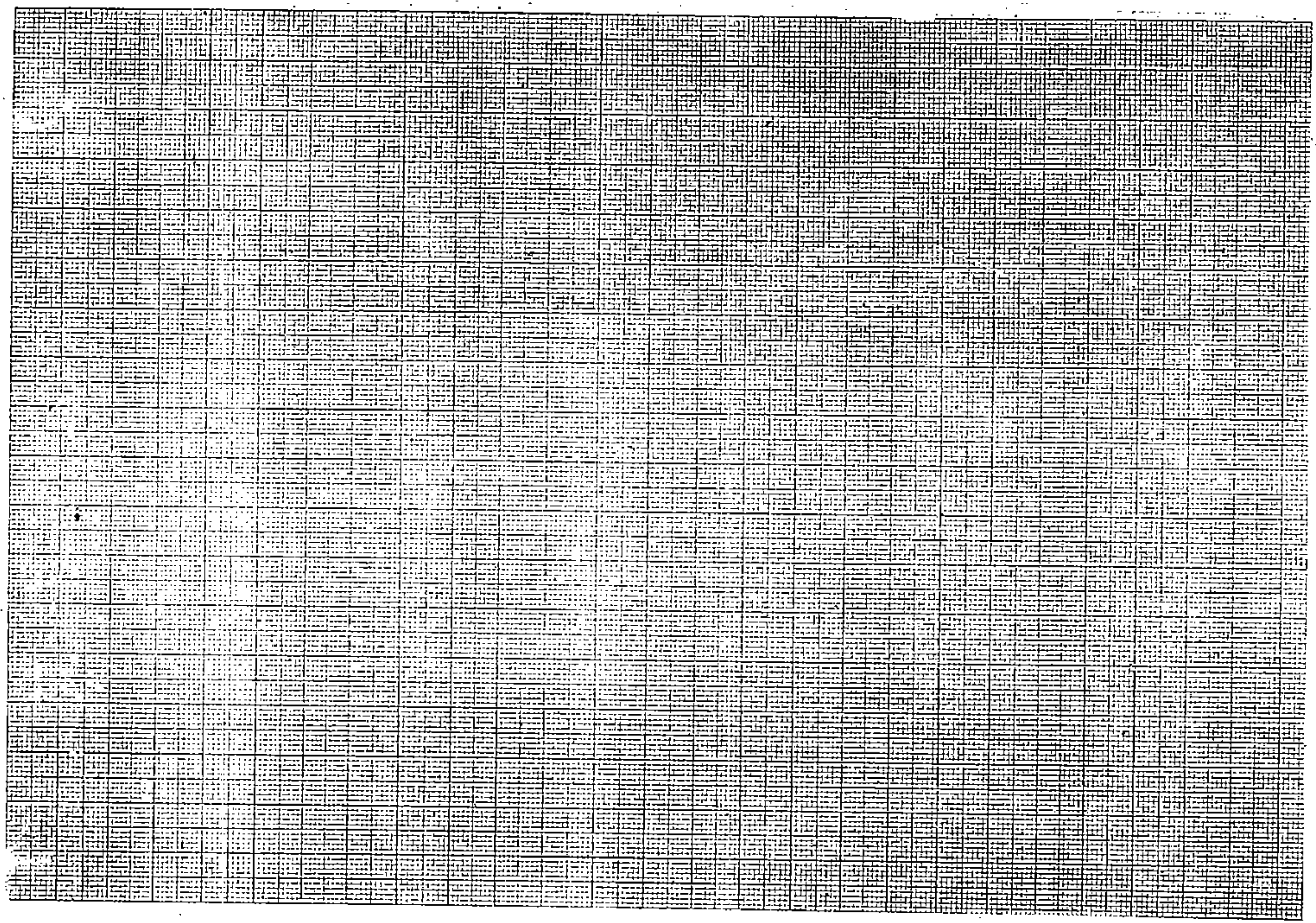
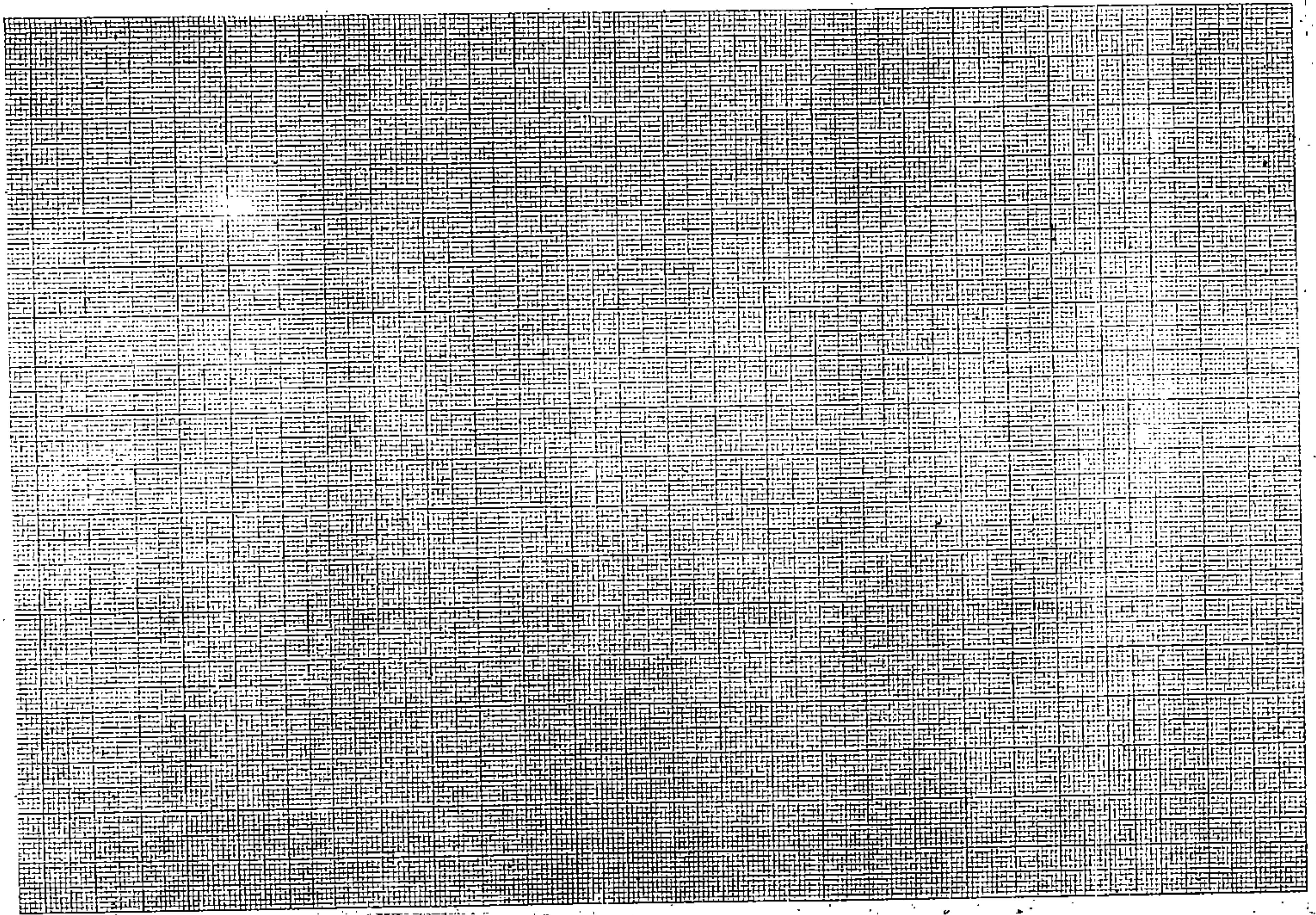


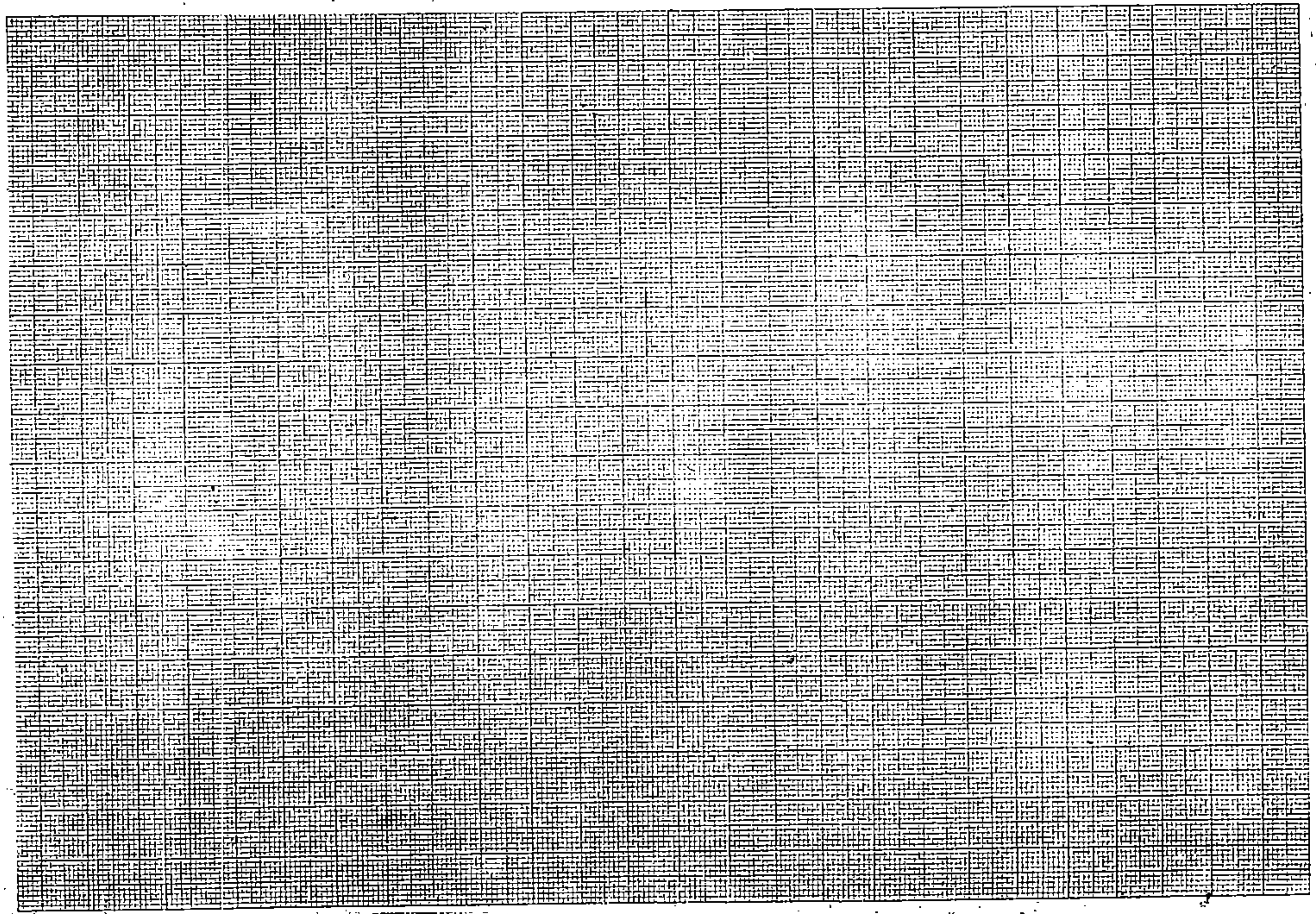
d)











UTTAR PRADESH
RAJARSHI TANDON OPEN UNIVERSITY

UGMM - 01
Calculus

Block

3

INTEGRAL CALCULUS

BLOCK 3 INTRODUCTION

This block and the next deal with integration. If you look up any book on calculus, you will find that differentiation is studied in the first half, and integration is studied in the second. But actually, this is not the order in which these two were discovered. Mathematicians were familiar with at least some aspects of integration right from the fourth century B.C. Differentiation, on the other hand, was discovered in the seventeenth century. The Fundamental Theorem of Calculus, which establishes the inverse relationship between differentiation and integration, was proved towards the end of the seventeenth century. We shall study this theorem in Unit 10. Units 11, 12 and 13 will acquaint you with various methods of integration. You should try to solve all the exercises as you go along. "Practice makes a man/woman perfect" applies literally to problems of integration!

UNIT 10 DEFINITE INTEGRAL

Structure

10.1 INTRODUCTION

We have seen in Unit 3 of Block 1 that one of the problems which motivated the concept of a derivative was a geometrical one — that of finding a tangent to a curve at a point. The concept of integration was also similarly motivated by a geometrical problem — that of finding the areas of plane regions enclosed by curves. Some recently discovered Egyptian manuscripts reveal that the formulas for finding the areas of triangles and rectangles were known even in 1800 B.C. Using these one could also find the area of any figure bounded by straight line segments. But no method for finding the area of figures bounded by curves had evolved till much later.

In the third century B.C. Archimedes was successful in rigorously proving the formula for the area of a circle. His solution contained the seeds of the present day integral calculus. But it was only later, in the seventeenth century, that Newton and Leibniz were able to generalise Archimedes' method and also to establish the link between differential and integral calculus. The definition of the definite integral of a function, which we shall give in this unit was first given by Riemann in 1854. In Unit 11, we will acquaint you with various methods of integration.

You have probably studied integration before. But in this unit we shall adopt a new approach towards integration. When you have finished the unit, you should be able to tie in our treatment with your previous knowledge.

Objectives

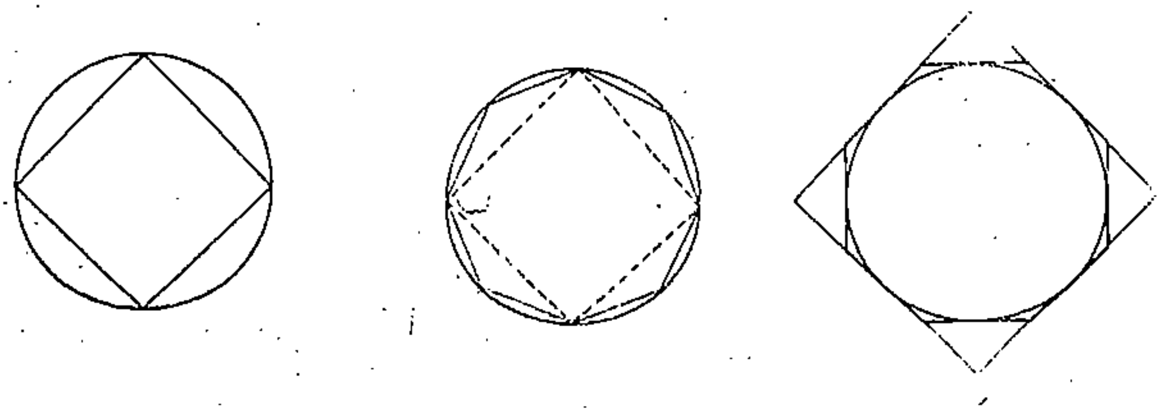
After reading this unit you should be able to

- define and calculate the lower and upper sums of some simple functions defined on [a,b], corresponding to a partition of [a,b],
- define the upper and lower integrals of a function,
- define the definite integral of a given function and check whether a given function is integrable or not,
- state and prove the Fundamental Theorem of Calculus,
- use the Fundamental Theorem to calculate the definite integral of an integrable function.

10.2 PRELIMINARIES

We have mentioned in the introduction that Archimedes was able to find the formula for the area of a circle. For this he approximated a circle by an inscribed regular polygon (See Fig. 1 (a)).

Further, we can see from Fig. 1(b) that this approximation becomes better and better as we increase the number of sides of the polygon. Archimedes also tried to approximate the area of the circle by a number of circumscribed polygons as in Fig. 1(c). The area of the circle was thus compressed between the inscribed and the circumscribed polygons.



(a) (b) (c)

Fig. 1

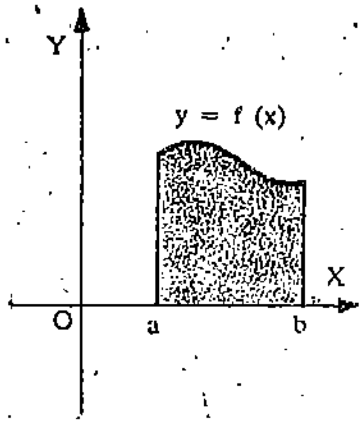


Fig. 2

By an ordered set we mean a set, in which, the order in which its elements occur is fixed.

We shall follow a similar procedure for finding the area of the shaded region shown in Fig. 2. We begin with the concept of a partition.

10.2.1 Partition of a Closed Interval

Let us consider the closed interval $[a,b] \subset R$. Then we have the following definitio

Definition 1 Let $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ be numbers in $[a,b]$ such that

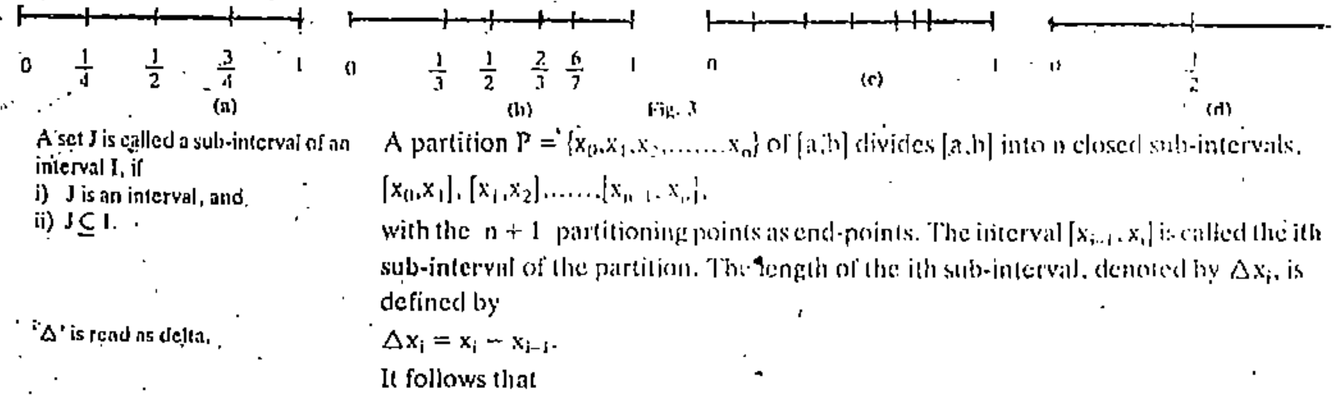
$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$.

Then the ordered set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is called a partition of $[a,b]$.

Example 1 $P_1 = \{0, 1/4, 1/2, 3/4, 1\}$ and $P_2 = \{0, 1/3, 1/2, 2/3, 6/7, 1\}$, both are partitions of $[0,1]$.

Moreover,

$P_1 \cup P_2 = \{0, 1/4, 1/3, 1/2, 2/3, 3/4, 6/7, 1\}$ and $P_1 \cap P_2 = \{0, 1/2, 1\}$ are also partitions of $[0,1]$. See Fig. 3 (a), (b), (c) and (d).



(a) (b) Fig. 3 (c) (d)

A set J is called a sub-interval of an interval I, if
i) J is an interval, and
ii) $J \subseteq I$.

A partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ divides $[a,b]$ into n closed sub-intervals,

$[x_0,x_1], [x_1,x_2], \ldots, [x_{n-1}, x_n]$,

with the $n+1$ partitioning points as end-points. The interval $[x_{i-1}, x_i]$ is called the ith sub-interval of the partition. The length of the ith sub-interval, denoted by Δx_i, is defined by

'Δ' is read as delta.

$\Delta x_i = x_i - x_{i-1}$.

It follows that

$$\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = b - a$$

We call partition P regular if every sub-interval has the same length, that is, if $x_1 - x_0$, $x_2 - x_1, \ldots, x_n - x_{n-1}$ are all equal. In this case, the length of $[a,b]$, that is $b-a$, is equally divided into n parts; and we get

$x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = \frac{b-a}{n}$.

Thus, a regular partition of $[a,b]$ may be written as

$\{a, a+h, a+2h, \ldots, a+nh\}$, where $a + nh = b$. We shall denote this partition by $\{a + ih\}_{i=0}^{n}$

For $P = \{1, 3/2, 2, 5/2, 3, 7/2, 4\}$, $\Delta x_1 = x_1 - x_0 = 3/2 - 1 = 1/2$, $\Delta x_2 = x_2 - x_1 = 2 - 3/2 = 1/2$. If you calculate Δx_3, Δx_4, Δx_5 and Δx_6, you will see that P is a regular partition of $[1,4]$.

E1) See Example 1. Which partitions among $P_1, P_2, P_1 \cup P_2$ and $P_1 \cap P_2$ are regular? What are the lengths of the third sub-intervals in P_1 and in P_2?

E2) Write down a regular partition for each of the following intervals.

a) $[0, 2]$ with 7 partitioning points.

b) $[2, 9]$ with 11 partitioning points.

Definition 2 Given two partitions P_1 and P_2 of $[a,b]$, we say that P_2 is a refinement of P_1 **(or P_2 is finer than P_1) if $P_2 \supset P_1$.**

In other words, P_2 is a refinement of P_1 if each sub-interval of P_2 is contained in some sub-interval of P_1.

Example 2 Consider the partitions

$P_1 = \{1, 5/4, 3/2, 7/4, 2\}$,

$P_2 = \{1, 6/5, 5/4, 3/2, 19/10, 2\}$,

$P_3 = \{1, 5/4, 3/2, 2\}$

P_1 and P_2 are both finer than P_3, as $P_1 \supset P_3$ and $P_2 \supset P_3$. However, neither is P_1 a refinement of P_2 nor is P_2 a refinement of P_1.

If P_1 and P_2 are partitions of $[a,b]$, then from Definition 2 it follows that

i) $P_1 \cup P_2$ is a refinement of both P_1 and P_2.

ii) P_1 and P_2 are both finer than $P_1 \cap P_2$.

Now, suppose for every $n \in N$ we define P_n as

$$P_n = \{a + i\frac{b-a}{2^n}\}_{i=0}^{2^n}$$

This means P_n has $2^n + 1$ elements. We can see that P_n is a regular partition, with each sub-interval having length $= \frac{b-a}{2^n}$.

$$\Delta x_i = x_i - x_{i-1} = a + i\frac{b-a}{2^n} - \{a + (i-1)\frac{b-a}{2^n}\} = \frac{b-a}{2^n}$$

Now, $\frac{b-a}{2^{n+1}} = \frac{1}{2}\left(\frac{b-a}{2^n}\right)$.

This means that the length of the sub-intervals corresponding to P_{n+1} is half the length of those corresponding to P_n. We can also see that $P_{n+1} \supset P_n$. In other words, P_{n+1} is finer than P_n (also see E3)). Thus, we have defined a sequence of partitions $\{P_n\}$ of $[a,b]$, such that P_{n+1} is a refinement of P_n for all n. Such a sequence $\{P_n\}$ is called a **sequence of refinements of partitions of $[a,b]$.**

E3) From the sequence of partitions $\{P_n\}$ defined above,

$P_1 = \{a, \frac{a+b}{2}, b\}$.

a) Find P_2 and P_3.

b) Verify that $P_3 \supset P_2 \supset P_1$.

c) What are the lengths of the sub-intervals in each of these partitions?

E4) Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[a,b]$, and let $P_1^* = P_1$, $P_2^* = P_1 \cup P_2$, $P_3^* = P_1 \cup P_2 \cup P_3$, and in general, $P_n^* = P_n \cup P_{n-1}^*$. Show that $\{P_n^*\}_{n=1}^{\infty}$ is a sequence of refinements of $[a,b]$.

10.2.2 Upper and Lower Product Sums

By now, we suppose you are quite familiar with partitions. Here we shall introduce the concept of product sums. It is through this that we shall be in a position to probe the more subtle concept of a definite integral in the next section.

Let $f: [a,b] \to R$ be a bounded function, and let
$P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$.

We have defined l.u.b. (supremum) and g.l.b. (infimum) of a bounded set of real numbers in Unit-1.

Now for any sub-interval $[x_{i-1}, x_i]$, consider the set $S_i = \{f(x) : x \in [x_{i-1}, x_i]\}$. Since f is a bounded function, S_i must be a bounded subset of **R**. This means, it has a supremum (or least upper bound) and infimum (or greatest lower bound). We write

$M_i = \sup S_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$, and
$m_i = \inf S_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$.

We now define the **upper product sum** $U(P,f)$ and the **lower product sum** $L(P,f)$ by

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i, \quad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \qquad \ldots\ldots (1)$$

You must have come across this Σ notation earlier. But let us state clearly what (1) means :

$U(P,f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \ldots + M_n(x_n - x_{n-1})$, and
$L(P,f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \ldots + m_n(x_n - x_{n-1})$.

G.F.B. Riemann (1826-1866)

Thus, to get $U(P,f)$ we have multiplied the supremum in each sub-interval by the length of that sub-interval, and have taken the sum of all such products. Similarly, $L(P,f)$ is obtained by summing the products obtained by multiplying the infimum in each sub-interval by the length of that sub-interval. $U(P,f)$ and $L(P,f)$ are also called **Riemann sums** after the mathematician George Friedrich Bernhard Riemann. Riemann gave a definition of definite integral that, to this day, remains the most convenient and useful one.

We started this unit saying that we wanted to find the area of the shaded region in Fig. 2. Then what are we doing with partitions, U(P,f) and L(P,f)? Fig. 4 will give you a clue to the path which we are going to follow to achieve our aim.

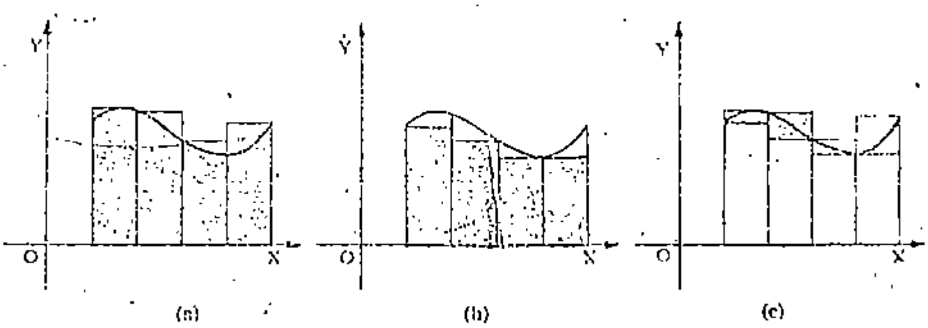


Fig. 4

Fig. 4(a) and 4(b) give the geometric view of $M_i \Delta x_i$ and $m_i \Delta x_i$ as areas of rectangles with base Δx_i and heights M_i and m_i, respectively.

The shaded rectangles in Fig. 4(a) are termed as outer rectangles, while the shaded rectangles in Fig. 4(b) are called inner rectangles.

Thus, when f is a non-negative valued function $(f(x) \geq 0 \ \forall \ x)$,

$U(P,f)$ = sum of the areas of outer rectangles as in Fig. 4(a).

$L(P,f)$ = sum of the areas of inner rectangles as in Fig. 4(b), and

$U(P,f) - L(P,f)$ = sum of the areas of the shaded rectangles along the graph of f in Fig. 4(c).

As you see from Fig. 5, $U(P,f)$ and $L(P,f)$ depend upon the function $f:[a,b] \to \mathbf{R}$ (compare Fig. 5(a) and (b)), and the partition P of $[a,b]$ (compare Fig. 5(c) and (d)).

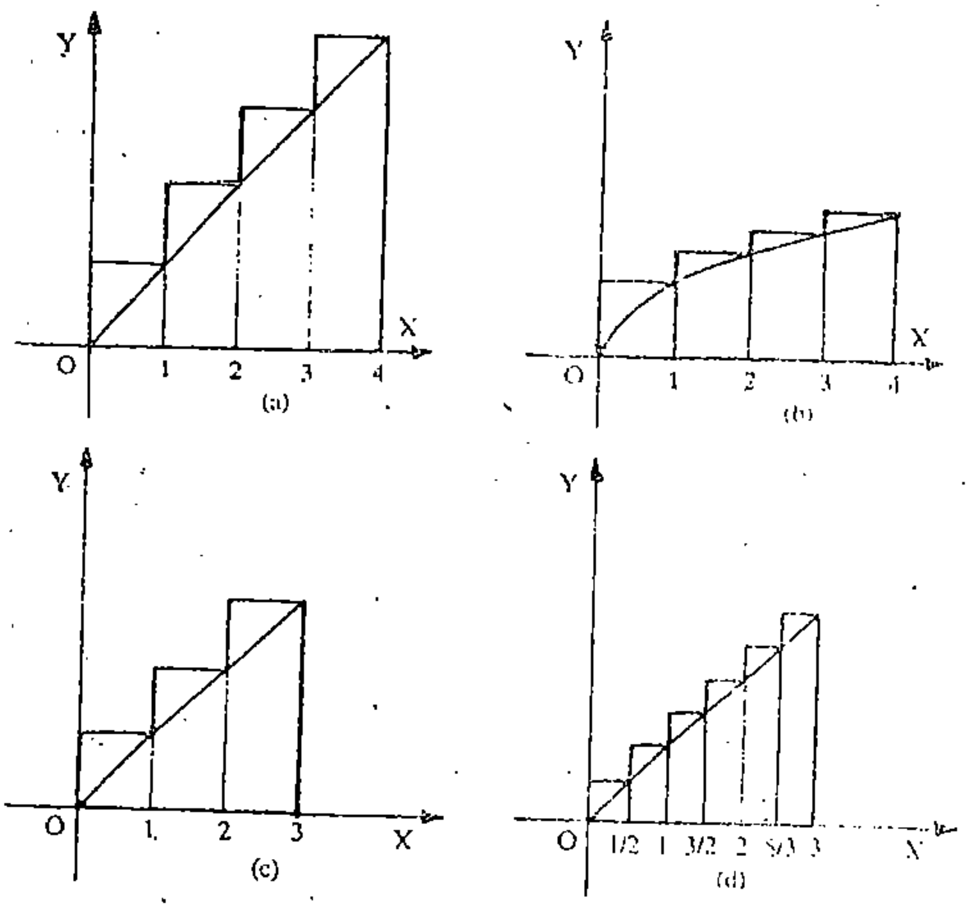


Fig. 5: (a) $U(P,f)$ where $y = x$ (b) $U(P,f)$ where $y^2 = x$
(c) $U(P,f)$ when $P = \{0, 1, 2, 3\}$ (d) $U(P,f)$ when $P = \{0, 1/2, 1, 3/2, 2, 5/2, 3\}$

If we denote the area between the curve given by $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, (the shaded area in Fig. 2) by A, then it is also quite clear from Fig. 4(a) and (b), that $L(P,f) \leq A \leq U(P,f)$.

The geometric view suggests the following theorem:

Theorem 1 Let $f:[a,b] \to \mathbf{R}$ be a bounded function, and let P be a partition of $[a,b]$. If M and m are the supremum and the infimum of f, respectively, in $[a,b]$, then

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a).$$

Proof: Now $M = \text{Sup } \{f(x) : x \in [a,b]\}$, and

$M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$. Hence $M_i \leq M$.

If $X \subset Y$, then $\sup X \leq \sup Y$, and $\inf X \geq \inf Y$

Further, $m = \inf \{f(x) : x \in [a,b]\}$, and

$m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$. Thus, $m \le m_i$. This means

$m \le m_i \le M_i \le M$ ...... (2)

Once we have the inequalities (2), we can complete our proof in easy steps. (2) implies that

$m \, \Delta x_i \le m_i \, \Delta x_i \le M_i \, \Delta x_i \le M \, \Delta x_i$

This implies that if we take the sum over $i = 1, 2, \ldots, n$, we get

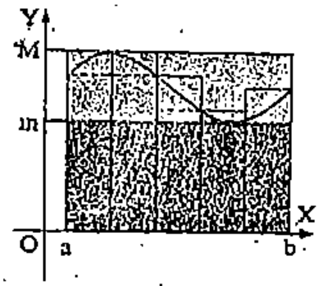


Fig. 6

$$m \sum_{i=1}^{n} \Delta x_i \le L(P,f) \le U(P,f) \le M \sum_{i=1}^{n} \Delta x_i$$

$$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a),$$

since $\sum_{i=1}^{n} \Delta x_i$ = the sum of the lengths of all sub-intervals

= the length of $[a,b]$

$= b-a$.

Fig. 6 will help you understand this theorem better. Let us verify this theorem in the case of a given function.

Example 3 Let $f:[1,2] \to \mathbb{R}$ be a function defined by $f(x) = x^2$, and let $P = \{1, 5/4, 3/2, 5/3, 2\}$ be a partition of $[1,2]$. The sub-intervals associated with P are $[1, 5/4]$, $[5/4, 3/2]$, $[3/2, 5/3]$ and $[5/3, 2]$.

The function f is a bounded function on $[1,2]$. In fact, the image set of f is $[1,4]$, which is obviously bounded.

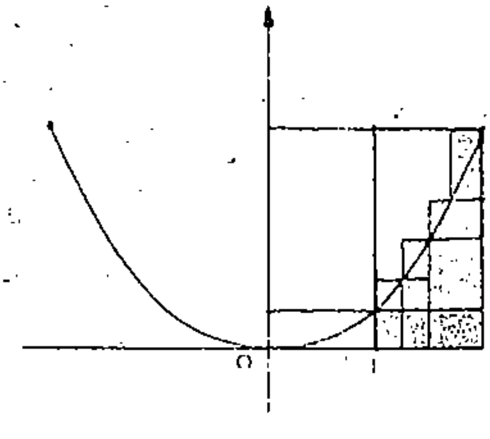


Fig. 7

Since f is an increasing function on each sub-interval (see Fig. 7) the supremum of f in $[x_{i-1}, x_i]$ will be attained at x_i and the infimum will be attained at x_{i-1}. That is,

$M_i = f(x_i)$ and

$m_i = f(x_{i-1})$. Therefore, we can write

$$U(P,f) = \Sigma M_i \, \Delta x_i = \Sigma f(x_i) \, \Delta x_i = \Sigma x_i^2 (x_i - x_{i-1})$$

$$= x_1^2 (x_1 - x_0) + x_2^2 (x_2 - x_1) + x_3^2 (x_3 - x_2) + x_4^2 (x_4 - x_3)$$

$$\therefore U(P,f) = \left(\frac{5}{4}\right)^2 \left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)^2 \left(\frac{1}{4}\right) + \left(\frac{5}{3}\right)^2 \left(\frac{1}{6}\right) + (2)^2 \left(\frac{1}{3}\right)$$

$$= \frac{25}{64} + \frac{9}{16} + \frac{25}{54} + \frac{4}{3}$$

$$= \frac{4751}{1728}$$

$L(P,f) = \Sigma m_i \Delta x_i = \Sigma f(x_{i-1}) \Delta x_i$

$$L(P,f) = (1)^2 \left(\frac{1}{4}\right) + \left(\frac{5}{4}\right)^2 \left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{3}\right)^2 \left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{25}{64} + \frac{9}{24} + \frac{25}{27}$$

$$= \frac{3652}{1728}$$

Now, the supremum of $f(x)$ in $[1,2] = M = f(2) = 2^2 = 4$, and
the infimum $= m = f(1) = 1$. Thus,
$M(b-a) = (4)(2-1) = 4$, and $m(b-a) = (1)(2-1) = 1$. Thus,
$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$.

We have noted that the upper and lower product sums depend on the partition of the given interval. Here we have a theorem which gives us a relation between the lower and upper sums corresponding to two partitions of an interval.

Theorem 2 Let $f:[a,b] \to R$ be a bounded function, and let P_1 and P_2 be partitions of $[a,b]$. If P_2 is finer than P_1, then
$L(P_1,f) \le L(P_2,f) \le U(P_2,f) \le U(P_1,f)$.

Proof: For proving this theorem we look at Fig. 8(a) and (b).
Let $P_1 = \{x_0, x_1, x_2, \ldots, x_n\}$ and $P_2 = \{x_0, v_1, x_1, x_2, \ldots, x_n\}$ be two partitions of $[a,b]$. P_2 contains one element more than P_1, namely, v_1.
Therefore, P_2 is finer than P_1.
In fact, P_2 can be rightly called a simple refinement of P_1. We shall prove the theorem for this simple refinement here.
P_1 divides $[a,b]$ into n sub-intervals ;
$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.

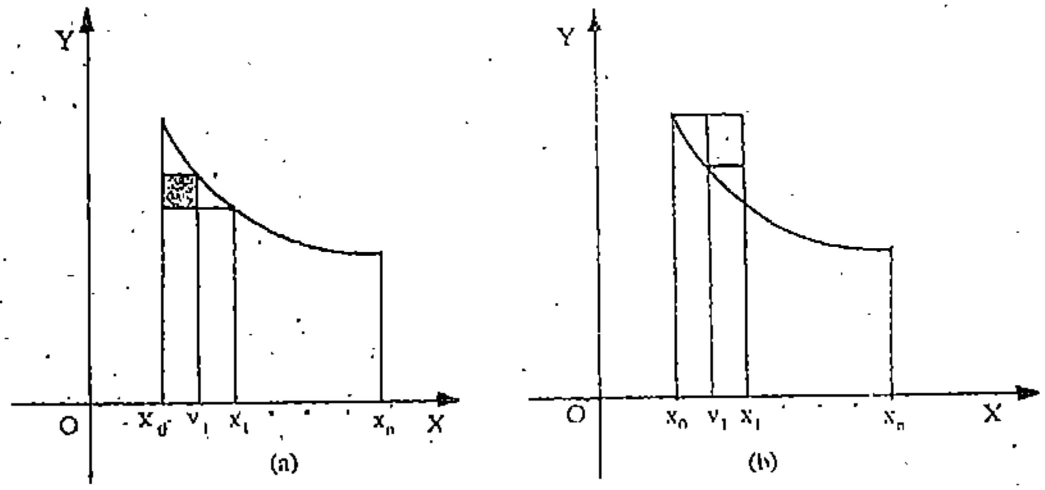


Fig. 8

Fig. 8(a) clearly shows that $L(P_1,f) \le L(P_2,f)$ (by an amount represented by the area of the shaded rectangle).
Similarly, Fig 8(b) shows that $U(P_2,f) \le U(P_1,f)$.
Since $L(P_2,f) \le U(P_2,f)$, the conclusion of the theorem follows in this case.

Now if P_2 is not a simple refinement of P_1, then suppose P_2 has m elements more than P_1. Then we can find $(m-1)$ partitions $P_2^1, P_2^2, P_2^3, \ldots\ldots, P_2^{m-1}$ such that

$P_1 \subset P_2^1 \subset P_2^2 \subset P_2^3 \subset \ldots\ldots \subset P_2^{m-1} \subset P_2$ and each partition in this sequence is a simple refinement of the previous one.

Theorem 2 then holds for each pair of successive refinements, and we get

$L(P_1,f) \le L(P_2^1,f) \le L(P_2^2,f) \le \ldots\ldots \le L(P_2^{m-1},f) \le L(P_2,f)$ and

$U(P_2,f) \le U(P_2^{m-1},f) \le \ldots\ldots \le U(P_2^2,f) \le U(P_2^1,f) \le U(P_1,f)$

Thus, $L(P_1,f) \le L(P_2,f) \le U(P_2,f) \le U(P_1,f)$.

From Theorem 2 we conclude the following :

Let $f:[a,b] \to \mathbf{R}$ be a continuous and non-negative valued function, and let $\{P_n\}_{n=1}^{\infty}$ be a sequence of refinements of $[a,b]$.

Then we have

$L(P_1,f) \le L(P_2,f) \le \ldots\ldots \le L(P_n,f) \le \ldots\ldots \le A \le \ldots\ldots \le U(P_n,f) \le \ldots\ldots \le U(P_2,f) \le U(P_1,f)$, where A is the area bounded by the curve, the x-axis and the lines $x = a$ and $x = b$.

E E5) Find the upper product sum and the lower product sum of the function f relative to the partition P, when

a) $f(x) = 1 + x^2$, $P = \{0, 1/2, 1, 3/2, 2\}$

b) $f(x) = 1/x$, $P = \{1, 2, 3, 4\}$

E E6) Verify Theorem 2 for the function $f(x) = 1/x$, $2 \le x \le 3$, and the partitions $P_1 = \{2, 5/2, 3\}$ and $P_2 = \{2, 9/4, 5/2, 11/4, 3\}$ of $[2,3]$.

In this sub-section we have seen that the area A in Fig. 2 can be approximated by means of the lower and upper sums corresponding to some partition of [a,b]. Further, Theorem 2 tells us that as we go on refining our partition, the lower and upper sums approach A from both sides. The lower sums underestimate A ($L(P,f) \le A$), while the upper sums overestimate A ($U(P,f) \ge A$). Let us go a step further in the next sub-section, and define lower and upper integrals.

10.2.3 Upper and Lower Integrals

Let $f:[a,b] \to \mathbf{R}$ be a non-negative bounded function. Then to each partition P of [a,b], there correspond the upper product sum $U(P,f)$ and the lower product sum $L(P,f)$.

Let $\mathcal{P}$ be the set of all partitions of [a,b]. Then the set $u = \{U(P,f) : P \in \mathcal{P}\}$ is a subset of $\mathbf{R}$ and is bounded below since $A \le U(P,f) \; \forall \; P \in \mathcal{P}$. Thus, it is possible to find the infimum of u.

Recall (Unit 1) that every set which is bounded below has an infimum, and every set which is bounded above has a supremum.

Similarly, the set $u' = \{L(P,f) : P \in \mathcal{P}\}$ is bounded above, since $L(P,f) \le A \; \forall \; P \in \mathcal{P}$. Hence we can find the supremum of u'. The infimum of u and the supremum of u' are given special names as you will see from this definition.

Definition 3 If a function f is defined on [a,b] and if $\mathcal{P}$ denotes the set of all partitions of [a,b], then infimum of $\{U(P,f) : P \in \mathcal{P}\}$ is called the upper integral of f

on [a,b], and is denoted by $\overline{\int_a^b} f(x)\, dx$.

The symbol '$\int$' is read as integral.

The supremum of $\{L(P,f) : P \in \mathcal{P}\}$ is called the lower integral of f on [a,b], and is

denoted by $\underline{\int_a^b} f(x)\, dx$.

From Theorem 2 it follows that $\overline{\int_a^b} f(x)\, dx \ge A$ and $\underline{\int_a^b} f(x)\, dx \le A$.

Thus we have $\underline{\int_a^b} f(x)\, dx \le A \le \overline{\int_a^b} f(x)\, dx$.

Example 4 Let us find $\overline{\int_0^1} f(x)\, dx$ and $\underline{\int_0^1} f(x)\, dx$.

for the function f, defined by $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$

Suppose $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of [0,1].

Each sub-interval $[x_{i-1}, x_i]$ contains both rational and irrational numbers. This means, $M_i = 1$ and $m_i = 0$ for each i.

Thus,

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} (1)(x_i - x_{i-1}) = 1 - 0 = 1.$$

and

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (0)(x_i - x_{i-1}) = 0.$$

Since P was any arbitrary partition of [0,1], this means that

$U(P,f) = 1$ and $L(P,f) = 0 \; \forall \; P \in \mathcal{P}$.

Thus, $u = \{U(P,f) : P \in \mathcal{P}\} = \{1\}$

and $u' = \{L(P,f) : P \in \mathcal{P}\} = \{0\}$

Hence inf $u = 1$ and sup $u' = 0$. That is,

$$\overline{\int_0^1} f(x)\, dx = 1 \text{ and } \underline{\int_0^1} f(x)\, dx = 0.$$

See if you can do these exercises now.

E7) Find $\overline{\int_0^1} f(x)\,dx$ and $\underline{\int_0^1} f(x)\,dx$, for the function f defined as $f(x) = 2$.

E8) If the functions f and g are bounded non-negative valued functions in [a,b] and if $f(x) \le g(x)$ in [a,b], prove that $\underline{\int_a^b} f(x)dx \le \underline{\int_a^b} g(x)dx$ and $\overline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} g(x)\,dx$.

10.3 DEFINITE INTEGRAL

In the last section we had restricted our discussion to non-negative valued functions. But we can easily extend our definitions of L(P,f), U(P,f) and the lower and upper integrals to all bounded functions. However, we shall have to modify our interpretation of these sums as areas. For this purpose, we introduce the concept of signed area. If R is any region, its signed area is defined to be the area of its portion lying above the x-axis, minus the area of its portion lying below the x-axis (see Fig. 9).

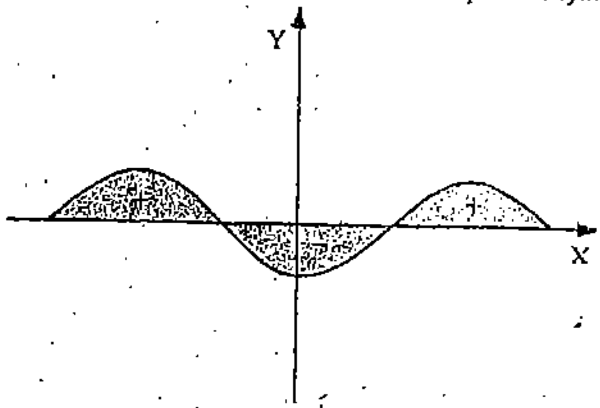


Fig. 9

With this definition then, we can interpret $L(P,f)$ as the signed area of a polygon inscribed inside the given region, and $U(P,f)$ as the signed area of a polygon circumscribed about the region. Thus, for any bounded function on a closed interval $[a,b]$, we can define $\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \in \mathcal{P}\}$, and

$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \in \mathcal{P}\}$

Now we are in a position to discuss the definite integral for a bounded function on a closed interval. (The adjective 'definite' anticipates the study of indefinite integral later).

Definition 4 Let $f:[a,b] \rightarrow \mathbb{R}$ be a bounded function. f is said to be **integrable** over $[a,b]$ if, and only if,

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx,$$

This common value is called the definite integral of f over the interval of integration $[a,b]$, and is denoted by $\int_a^b f(x)\,dx$.

In this notation for the definite integral, $f(x)$ is called the integrand, a is called the **lower limit** and b is called the **upper limit** of integration.

The symbol dx following $f(x)$ indicates the independent variable. Here x is merely a dummy variable, and we may replace it by t or v, or any other letter. This means,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(v)\,dv.$$

The symbol $\int$ reminds us of S which is appropriate, because a definite integral is, in some sense, the limit of a sum. In fact it is the common value (when it exists) of the lower and upper integrals which are themselves infimum and supremum sums.

The use of $f(x)\,dx$ reminds us that we do not take the sum of function values. Rather we take the sum of terms, each of which is the product of the supremum or infimum of the function in an interval multiplied by the length of the sub-interval

The definition of definite integral above, applies only if $a < b$, but it would be appropriate to include the cases $a = b$ and $a > b$ as well. In such cases we define

$$\int_a^a f(x)\,dx = 0$$

and $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

provided the right hand integral exists.

In Example 4, we have seen that if

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational, then} \end{cases}$$

$$\underline{\int_0^1} f(x)\,dx = 0, \text{ and } \overline{\int_0^1} f(x)\,dx = 1.$$

Since the lower and upper integrals for this function are not equal, we conclude that it is not integrable.

E9) Check whether the function given in E7) is integrable or not.

Now we shall list some basic properties of definite integrals.

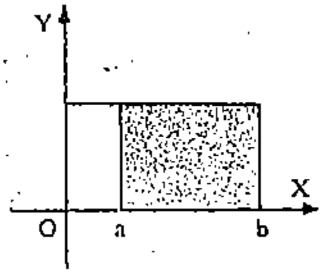


Fig. 10

I Integral of a constant function f(x) = c

$$\int_a^b c\,dx = c(b-a)$$

This is intuitively obvious since the area represented by the integral is simply a rectangle with base $b-a$ and height c (see Fig. 10.).

Now let us consider a function f which is integrable over [a, b].

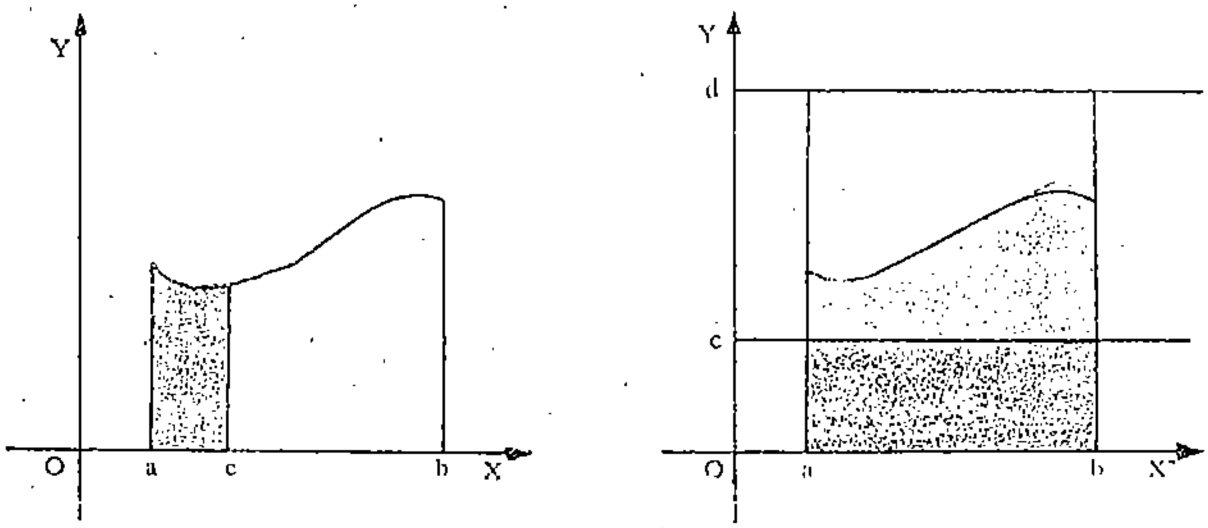
II Constant Multiple Property

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx.$$

III Interval Union Property

If $a < c < b$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Its geometrical interpretation is shown in Fig. 11(a).



(a) (b)

Fig. 11

IV Comparison Property

If c and d are constants such that $c \le f(x) \le d$ for all x in [a,b], then

$$c(b-a) \le \int_a^b f(x)\,dx \le d(b-a)$$

Fig. 11(b) makes this statement clearer. Note that c and d are not necessarily the minimum and maximum values of f(x) on [a,b]. c may be less than the minimum, and d may be greater than the maximum.

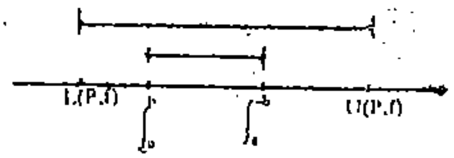
The following theorem given a criterion for a function to be integrable.

Theorem 3 A bounded function f is integrable over [a,b] if and only if, for every $\varepsilon > 0$, there exists a partition P of [a,b] such that $0 \le U(P,f) - L(P,f) < \varepsilon$.

Proof : We know that for any partition P of [a,b],

$$L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f)$$

$$\Rightarrow 0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(P,f) - L(P,f).$$



If the function f has the property that for every $\varepsilon > 0$ there exists a partition P of [a,b] such that
$U(P,f) - L(P,f) < \varepsilon$, we conclude that

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon \text{ for every } \varepsilon > 0.$$

From this it follows that $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0$ and hence f is integrable over [a,b].

On the other hand, if f is integrable over [a,b],

$\int_a^b f(x)\,dx = \sup\{L(P,f) : P \in \mathcal{P}\} = \inf\{U(P,f) : P \in \mathcal{P}\}$. Thus, for every $\varepsilon > 0$ we can find partitions P′ and P″ of [a,b], such that

$0 \le \int_a^b f(x)\,dx - L(P',f) < \varepsilon/2$, and $0 \le U(P'',f) - \int_a^b f(x)\,dx < \varepsilon/2$ (see Sec. 2 of Unit 1).

Taking some partition P which is finer than both P′ and P″, and adding the two inequalities, we have

$0 \le U(P,f) - L(P,f) < \varepsilon$.

This completes the proof.

Now arises a natural question : Which are the functions which satisfy the above criterion? The following theorems provide an answer.

Theorem 4 A function that is monotonic (increasing or decreasing) on [a,b], is integrable over [a,b].

Proof Let the function $f:[a,b] \to \mathbf{R}$ be increasing. Then
$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$.

For each positive integer n, let $P_n = \{a, a+h, \ldots, a+nh = b\}$, where $h = \frac{b-a}{n}$, be a regular partition of [a,b]. Then

$$U(P_n,f) = \sum_{i=1}^{n} M_i\,\Delta x_i = \sum_{i=1}^{n} M_i h = h\sum_{i=1}^{n} M_i = \frac{b-a}{n}\sum_{i=1}^{n} f(a+ih)$$

since the supremum of f(x) in $[a+(i-1)h, a+ih]$ is $f(a+ih)$.

and $L(P_n,f) = h\sum_{i=1}^{n} m_i = \frac{b-a}{n}\sum_{i=1}^{n} f(a+(i-1)h)$

Therefore,

$$U(P_n,f) - L(P_n,f) = \frac{b-a}{n}\,[f(a+h) + f(a+2h) + \dots + f(a+nh) - f(a) - f(a+h) \dots f(a+(n-1)h)]$$

$$= \frac{b-a}{n}\,[f(a+nh) - f(a)]$$

$$= \frac{b-a}{n}\,[f(b) - f(a)].$$

Let $\varepsilon > 0$. Can we choose an n which will make $U(P_n,f) - L(P_n,f) < \varepsilon$?

$n > \frac{(b-a)[f(b)-f(a)]}{\varepsilon}$

$\Rightarrow h = \frac{b-a}{n} < \frac{\varepsilon}{f(b)-f(a)}$

Yes, we can. Try some $n > \frac{(b-a)\,[f(b) - f(a)]}{\varepsilon}$. If we substitute this value of n in [1], we get

$$U(P_n,f) - L(P_n,f) < \frac{\varepsilon\,(b-a)\,[f(b) - f(a)]}{(b-a)\,[f(b) - f(a)]} < \varepsilon$$

Thus, applying Theorem 3, we can conclude that f is integrable.

Theorem 4 leads us to the following useful result.

Corollary 1 If f is increasing or decreasing on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

$$= \lim_{h\to 0} h[f(a+h) + f(a+2h) + \dots + f(a+nh)], \text{ where } h = \frac{b-a}{n}.$$

We shall illustrate the usefulness of Corollary 1 through some examples. But before that we state another theorem, which identifies one more class of integrable functions.

Theorem 5 If a function $f:[a,b] \to \mathbb{R}$ is continuous, then f is integrable.

The proof of this theorem is beyond the scope of this course. We shall prove it in a later course on real analysis.

In Sec. 5 in Unit 3, we have seen that differentiability implies continuity. Now we can write

differentiability $\Rightarrow$ continuity $\Rightarrow$ integrability

Now, let us evaluate some definite integrals with the help of Corollary 1.

Example 5 To evaluate $\int_a^b \cos x\,dx$, $0 < a < b < \pi/2$, we observe that $f : x \to \cos x$ is a decreasing function on $[a,b]$. Therefore, by Corollary 1

$$\int_a^b \cos x\,dx = \lim_{h\to 0} h[\cos(a+h) + \cos(a+2h) + \dots + \cos(a+nh)],\ a+nh = b.$$

Now

$$2\sin(h/2)\,[\cos(a+h) + \cos(a+2h) + \dots + \cos(a+nh)]$$

$$= 2\sin(h/2)\cos(a+h) + 2\sin(h/2)\cos(a+2h) + \dots + 2\sin(h/2)\cos(a+nh).$$

$$= [\sin(a+\tfrac{3h}{2}) - \sin(a+\tfrac{h}{2})] + [\sin(a+\tfrac{5h}{2}) - \sin(a+\tfrac{3h}{2})] + \dots + [\sin(a+(\tfrac{2n+1}{2})h) - \sin(a+(\tfrac{2n-1}{2})h)]$$

$$= \sin(a+(\tfrac{2n+1}{2})h) - \sin(a+\tfrac{h}{2})$$

$$= \sin(b+\tfrac{h}{2}) - \sin(a+\tfrac{h}{2}), \text{ since } a+nh = b$$

$$\Rightarrow \cos(a+h) + \cos(a+2h) + \dots + \cos(a+nh) = \frac{\sin(b+h/2) - \sin(a+h/2)}{2\sin h/2}$$

Thus,

$$\int_a^b \cos x\,dx = \lim_{h\to 0}\,[\sin(b+\tfrac{h}{2}) - \sin(a+\tfrac{h}{2})]\cdot\frac{h/2}{\sin(h/2)}$$

$$= \sin b - \sin a.$$

Example 6 Suppose we want to evaluate $\int_1^2 (x+x^2)\,dx$

Here, $f : x \rightarrow x + x^2$ is an increasing function on $[1,2]$.

Therefore,

$$\int_1^2 (x+x^2)\,dx = \lim_{h\to 0} h \sum_{i=1}^{n} f(1 + ih),\ h = 1/n$$

$$= \lim_{h\to 0} h \sum_{i=1}^{n} [(1+ih) + (1+ih)^2]$$

$$= \lim_{h\to 0} h \sum_{i=1}^{n} (2 + 3hi + h^2 i^2)$$

$$= \lim_{h\to 0} [2h \sum_{i=1}^{n} 1 + 3h^2 \sum_{i=1}^{n} i + h^3 \sum_{i=1} i^2]$$

$$= \lim_{h\to 0} [2nh + \frac{3}{2} h^2 n(n+1) + \frac{1}{6} h^3 n(n+1)(2n+1)]$$

$$= \lim_{h\to 0} [2 + \frac{3}{2}(1 + h) + \frac{1}{6}(1+h)(2+h)],\ \text{since } nh = 1.$$

$$= 2 + \frac{3}{2} + \frac{1}{3} = \frac{23}{6}.$$

Recall that

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

In this section we have noted that a continuous function is integrable. We have also proved that a monotone function is integrable. Corollary 1 gives us a method of finding the integral of a monotone function. One condition which is very essential for the integrability of a function in an interval, is its boundness in that interval. If a function is unbounded, it cannot be integrable. In fact, if a function is not bounded, we cannot talk of M_i or m_i, and thus cannot form the upper or lower product sums. Now on the basis of the criteria discussed in this section you should be able to solve this exercise.

E10) State whether or not each of the following functions is integrable in the given interval. Give reasons for each answer.

a) $f(x) = x^2 - 2x + 2$ in $[-1,5]$

b) $f(x) = \sqrt{x}$ in $[1,a]$

c) $f(x) = 1/x$ in $[-1,1]$

d) $f(x) = [x]$ in $[0,4]$

e) $f(x) = |x-1|$ in $[0,3]$

f) $f(x) = \frac{x^2 + 1}{2x + 1}$ in $[-4,0]$

g) $f(x) = \begin{cases} x + 1 \text{ when } x < 0 \\ 1 - x \text{ when } x \geq 0, \end{cases}$ in $[-1,1]$

h) $f(x) = \begin{cases} x + 1 \text{ when } x < 1 \\ 2x + 1 \text{ when } x \geq 1, \end{cases}$ in $[0,3]$

E11) Use Corollary 1 to evaluate the following definite integral.

$$\int_0^2 (1+x)\, dx$$

10.4 FUNDAMENTAL THEOREM OF CALCULUS

As you have already read in the introduction, the basic concepts of definite integral were used by the ancient Greeks, mainly Archimedes (287-212 B.C.), more than 2000 years ago. This was long before calculus was invented. But in the seventeenth century Newton and Leibniz developed a procedure for evaluating a definite integral by antidifferentiation. This procedure is embodied in the Fundamental Theorem of Calculus (FTC).

Before we state this theorem, we introduce the notions of the average value of a function and the antiderivative of a function.

Definition 5 Let f be integrable over [a,b]. The average value $\bar{y}$ of $f(x)$ over [a,b] is

$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

The following theorem tells us that every continuous function on a closed interval attains its average value at some point of the interval. We shall not give its proof here.

Theorem 6 (Average Value Theorem) If $f : [a,b] \to \mathbf{R}$ is continuous, then

$$f(\bar{x}) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

for some $\bar{x} \in [a,b]$.

We shall now define the antiderivative of a function.

Definition 6 Let $f : [a,b] \to \mathbf{R}$ and $F : [a,b] \to \mathbf{R}$ be two functions such that $\frac{d}{dx}(F(x)) = F'(x) = f(x)$ for each $x \in [a,b]$. We call $F(x)$ an antiderivative (or inverse derivative) of $f(x)$.

For example, $\frac{x^3}{3}$ is an antiderivative of x^2, since $\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$;
$-\cos x$ is an antiderivative of $\sin x$, since $\frac{d}{dx}(-\cos x) = \sin x$.
Is $\frac{x^4}{4} - x^2$ an antiderivative of $x^3 - 2x$?

Consider the two functions $f(x) = x^2$ and $g(x) = x^2 + 5$. Both these are antiderivatives of the function $h(x) = 2x$. This means that antiderivative of a function is not unique. In fact, if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + c$ is also an antiderivative of $f(x)$. This follows from the fact that

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}(F(x)+c) = f(x).$$

We can also say that any two antiderivatives of a function differ only by a constant. Because, if $F(x)$ and $G(x)$ are two antiderivatives of $f(x)$, then $F'(x) = G'(x) = f(x)$. That is, $[F(x) - G(x)]' = 0$.

We have noted in Unit 7 that if the derivative of a function is zero on an interval, then that function must be a constant. Thus $(F(x) - G(x)) = c$.

The interval $[a, b]$ on which f and its antiderivative are defined, so that $F'(x) = f(x)$ $\forall x \in]a, b[$, is implicit in our discussion here.

Now having defined the average value and the antiderivative, we are in a position to state the Fundamental Theorem of Calculus. We shall give this theorem in two parts.

Theorem 7 (FTC): Let $f : [a, b] \to R$ be a continuous function.

Part 1 If the function $F : [a,b] \to R$ is defined by

$$F(x) = \int_a^x f(t)\, dt, \qquad \text{.....(3)}$$

then F is an antiderivative of f, that is, $F'(x) = f(x)$ for all x in $]a,b[$.

Part 2 If G is an antiderivative of f in $]a,b[$, then

$$\int_a^b f(x)\, dx = G(x)\Big]_a^b = G(b) - G(a).$$

Proof of Part 1.

By the definition of derivative, $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)\, dt\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt,$$

by the interval union property of definite integrals.

But, by the Average Value Theorem (Theorem 6)

$\frac{1}{h}\int_x^{x+h} f(t)\, dt = f(\bar{t})$ for some $\bar{t} \in [x, x+h]$.

Therefore, $F'(x) = \lim_{h\to 0} f(\bar{t})$. We know that $\bar{t} \in [x,x+h]$. This means that as $h \to 0$, $\bar{t} \to x$. Therefore,

$F'(x) = \lim_{\bar{t}\to x} f(\bar{t}) = f(x)$, since f is a continuous function.

Hence, F as defined by (3), is an antiderivative of f.

Proof of Part 2

G is given as an antiderivative of f in $]a,b[$. Also, as shown in Part 1, F defined by (3) is an antiderivative of f in $]a,b[$. Therefore,

$G(x) = F(x) + c$ on $]a,b[$ for some constant c.

To evaluate c, we substitute $x = a$, and obtain

$c = G(a) - F(a) = G(a) - 0 = G(a)$.

$F(a) = \int_a^a f(t)\, dt = 0$

Hence $G(x) = F(x) + G(a)$, or

$F(x) = G(x) - G(a)$

If we put $x = b$, we get

$$F(b) = \int_a^b f(x)\, dx = G(b) - G(a)$$

The Fundamental Theorem of Calculus tells us that differentiation and integration are inverse processes, because Part 1 may be rewritten as

$$\frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x), \text{ if } f \text{ is continuous.}$$

That is, if we first integrate the continuous function f with the variable x as the upper limit of integration and then differentiate with respect to x, the result is the function f again. So differentiation offsets the effect of integration.

On the other hand, if we assume that G′ is continuous, then Part 2 of FTC may be written as

$$\int_a^x G'(t)\, dt = G(x) - G(a)$$

Here we can say that if we first differentiate the function G and then integrate the result from a to x, the result can differ from the original function G(x) only by the constant G(a). If G is so chosen that G(a) = 0, then integration offsets the effect of differentiation.

Till now we had evaluated the integrals of some functions by first finding the lower and upper sums, and then taking their supremum and infimum, respectively. This is a tedious procedure and we cannot apply it easily to all functions. But now, FTC gives us an easy method of evaluating definite integrals. We shall illustrate this through some examples.

Example 7 Suppose we want to evaluate $\int_2^3 (ax^2+bx+c)\, dx$.

Since $f : x \to ax^2 + bx + c$ is continuous on [2,3], it is integrable over [2,3].

$G(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$ is an antiderivative of f(x).

Hence, by FTC (Part 2)

$$\begin{aligned}\int_2^3 (ax^2 + bx + c)\, dx &= G(x)\Big]_2^3 \\ &= G(3) - G(2) \\ &= (9a + 9b/2 + 3c) - (8a/3 + 2b + 2c) \\ &= 19\frac{a}{3} + 5\frac{b}{2} + c\end{aligned}$$

Example 8 Let us evaluate $\int_0^{\pi/4} \cos 2x\, dx$

$$\begin{aligned}\int_0^{\pi/4} \cos 2x\, dx &= \frac{\sin 2x}{2}\Big]_0^{\pi/4} \\ &= \frac{\sin(\pi/2)}{2} - \frac{\sin 0}{2} \\ &= \frac{1}{2}\end{aligned}$$

Example 9 To evaluate $\frac{d}{dx}\int_0^{x^2} \sin t\, dt$, we put $x^2 = u$

Then

$$\begin{aligned}\frac{d}{dx}\int_0^{x^2} \sin t\, dt &= \frac{d}{dx}\int_0^{u} \sin t\, dt \\ &= \frac{d}{du}\left(\int_0^u \sin t\, dt\right) \frac{du}{dx}\end{aligned}$$

Now $\frac{du}{dx} = 2x$, and using FTC (Part 1), we get

$\frac{d}{du}\int_0^u \sin t\, dt = \sin u = \sin x^2$. Thus, $\frac{d}{dx}\int_0^{x^2} \sin t\, dt = 2x . \sin x^2$

Example 9 suggests the following formula.

$$\frac{d}{dx}\left(\int_0^{g(x)} f(t)dt\right) = f(g(x))g'(x).$$

If you have followed these examples, you should be able to solve the exercises below. Remember that the main thing in evaluating a definite integral is to find an antiderivative of the given function.

E **E12)** **The second column in the table below consists of some functions which are antiderivatives of the functions given in column 1. Match a function with its antiderivative by pairing appropriate numbers.**

For example, we can match x^n with $\frac{x^{n+1}}{n+1}$ since $\frac{x^{n+1}}{n+1}$ is an antiderivative of x^n. We shall indicate this by iii) → viii).

Function	**Antiderivative**
i) $\sin x$	i) $-\ln \cos x$
ii) $\cos x$	ii) $\ln \cosh x$
iii) x^n	iii) $\text{sech} x$
iv) e^{ax}	iv) $-\cos x$
v) $\tan x$	v) $\sin x$
vi) a	vi) $\frac{1}{a} e^{ax}$
vii) $\tanh x$	vii) ax
viii) $\text{sech} x \tanh x$	viii) $\frac{x^{n+1}}{n+1}$

E E13) Evaluate the following integrals by using FTC.

a) $\int_1^3 2x^3 dx$ b) $\int_1^3 (2x^2+2x+1)\, dx$

c) $\int_1^2 x(x+1)^2 dx$ d) $\int_0^{\pi/4} \sec^2 x\, dx$

e) $\int_0^2 x(x^2+1)^2\, dx$ f) $\int_{-\pi/2}^{\pi/2} (x+\sin x)\, dx$

g) $\int_0^{\pi} (x - \cos x)\, dx$ h) $\int_0^4 e^{2x}\, dx$

i) $\int_0^1 \sinh x \cosh x\, dx$ j) $\int_{-1}^1 (\sinh x + \cosh x)\, dx$

E14) Find $\frac{d}{dx}[F(x)]$ when $F(x)$ is defined by the following definite integrals.

a) $\int_0^x \sqrt{1+t^2}\,dt$

b) $\int_0^{x^2} \sqrt{\sin t + \cos t}\,dt$

c) $\int_0^{\sqrt{1-x^2}} (t^3 - 2t + 1)\,dt$

d) $\int_x^{x^2} \cos t^2\,dt$

e) $\int_{\sqrt{x}}^{x^2} t\sqrt{1-t^2}\,dt$

10.5 SUMMARY

In this unit we have covered the following points:

1) A partition P of a closed interval $[a,b]$ is a set $\{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$ such that $x_0 < x_1 < x_2 < \ldots < x_n$.

2) A partition P_1 of $[a,b]$ is finer than a partition P_2, if $P_1 \supseteq P_2$.

3) If M and m are the supremum and the infimum of a bounded function f in $[a,b]$, then, given any partition P of $[a,b]$, $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$.

4) The lower integral of a bounded function is less than or equal to its upper integral.

5) A bounded function f is integrable over $[a,b]$ if and only if its lower and upper integrals are equal. In such a situation the lower (or upper) integral is called the definite integral of f over $[a,b]$, denoted by $\int_a^b f(x)\,dx$.

6) If f is monotonic or continuous on $[a,b]$, then f is integrable over $[a,b]$.

7) If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx$ represents the signed area of the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

8) If f is monotonic on $[a,b]$, then

$$\int_a^b f(x)dx = \lim_{h\to 0} h \sum_{i=1}^{n} f(a + ih), \quad = \quad \lim_{h\to 0} h \sum_{i=1}^{n} f(a + (i-1)h),$$

where $h = \frac{b-a}{n}$

9) The Fundamental Theorem of Calculus:

i) If f is continuous on $[a,b]$, then for $x \in]a,b[$

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

ii) If f is continuous on $[a,b]$ and $F'(x) = f(x)$ for $x \in]a,b[$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

10.6 SOLUTIONS AND ANSWERS

E 1) P_1, $P_1 \cap P_2$ are regular.

$\Delta x_3 = 1/4$ in P_1, $\Delta x_3 = 1/6$ in P_2.

E 2) a) $\{0, \frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, 1\frac{2}{3}, 2\}$

b) $\{2, 2\frac{7}{10}, 3\frac{2}{5}, 4\frac{1}{10}, 4\frac{4}{5}, 5\frac{1}{2}, 6\frac{1}{5}, 6\frac{9}{10}, 7\frac{3}{5}, 8\frac{3}{10}, 9\}$

E 3) a) $P_2 = \{a, \frac{3a+b}{4}, \frac{a+b}{2}, \frac{a+3b}{4}, b\}$

$P_3 = \{a, \frac{7a+b}{8}, \frac{3a+b}{4}, \frac{5a+3b}{8}, \ldots, b\}$

c) Δx in P_2 is $\frac{b-a}{4}$.

Δx in P_3 is $\frac{b-a}{8}$.

E 4) $P^*_{n+1} = P_{n+1} \cup P^*_n \Rightarrow P^*_n \subset P^*_{n+1} \Rightarrow P^*_{n+1}$ is a refinement of $P^*_n \;\forall\; n$.

E 5) a) f(x) is an increasing function on [0,2]. Hence

$L(P,f) = 1\cdot\frac{1}{2} + \frac{5}{4}\cdot\frac{1}{2} + 2\cdot\frac{1}{2} + \frac{13}{4}\cdot\frac{1}{2}$

and $U(P,f) = \frac{5}{4}\cdot\frac{1}{2} + 2\cdot\frac{1}{2} + \frac{13}{4}\cdot\frac{1}{2} + 5\cdot\frac{1}{2}$

b) f(x) is a decreasing function on [1,4]. Hence

$L(P,f) = \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1 + \frac{1}{4}\cdot 1$

$U(P,f) = 1\cdot 1 + \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1$

E 6) $L(P_1, f) = \frac{2}{5}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = \frac{11}{30}$

$U(P_1, f) = \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{2} = \frac{9}{20}$

$L(P_2, f) = \frac{4}{9} \cdot \frac{1}{4} + \frac{2}{5} \cdot \frac{1}{4} + \frac{4}{11} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}$

$= \frac{2289}{5940}$

$U(P_2, f) = \frac{1}{2} \cdot \frac{1}{4} + \frac{4}{9} \cdot \frac{1}{4} + \frac{2}{5} \cdot \frac{1}{4} + \frac{4}{11} \cdot \frac{1}{4}$

$= \frac{1691}{3960}$

$L(P_1, f) \le L(P_2, f) \le U(P_2, f) \le U(P_1, f)$.

E 7) If $P = \{x_0, \ldots\ldots, x_n\}$ is a partition of $[0,1]$,
$L(P,f) = U(P,f) = \Sigma\, m_i \Delta x_i = \Sigma 2 \Delta x_i = 2 \Sigma \Delta x_i = 2 \cdot 1 = 2$

Hence $\underline{\int_0^1} f(x)\, dx = \overline{\int_0^1} f(x)\, dx = 2$.

E 8) If $P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$ is any partition of $[a,b]$,
then $L(P,f) = \Sigma m_{i,f}\, \Delta x_i \le \Sigma\, m_{i,g}\, \Delta x_i = L(P,g)$,
where $m_{i,f} = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and
$m_{i,g} = \inf\{g(x) : x \in [x_{i-1}, x_i]\}$
and $m_{i,f} \le m_{i,g}$ since $f(x) \le g(x)$ for all x.
Similarly, $U(P,f) \le U(P,g)$ for all P.
The result follows.

E 9) $f(x) = 2$ is integrable

E 10) a), b), e) and g) are integrable as these are continuous.
c), f) are not integrable as they are not bounded.
d), h) are integrable as these are increasing functions.

E 11) $\int_1^2 (1+x)\, dx = \lim_{h \to 0} h\, [2 + 2 + \frac{1}{n} + 2 + \frac{2}{n} + 2 + \frac{3}{n} + \ldots\ldots + 2 + \frac{n-1}{n}],\ h = \frac{1}{n}$

$= \lim_{n \to \infty} \frac{1}{n} [2n + \frac{1}{n}(1 + 2 + \ldots\ldots + n-1)]$

$= \lim_{n \to \infty} [2 + \frac{n(n-1)}{2n^2}] = 2 + \frac{1}{2} = \frac{5}{2}$

E 12) i) → iv)
ii) → v)
iii) → viii)
iv) → vi)
v) → i)
vi) → vii)
vii) → ii)
viii) → iii)

E 13) a) $\frac{x^4}{2}$ is an antiderivative of $2x^3$. Hence

$\int_1^3 2x^3\, dx = \frac{x^4}{2}\Big|_1^3 = \frac{81}{2} - \frac{1}{2} = 40$

b) $\int_1^3 (2x^2 + 2x + 1)\, dx = \frac{2x^3}{3} + x^2 + x\Big]_1^3$

$= 18 + 9 + 3 - (\frac{2}{3} + 1 + 1) = 27\frac{1}{3}$

c) $\frac{119}{12}$, d) 1, e) $\frac{62}{3}$, f) 0, g) $\frac{\pi^2}{2}$

h) $\frac{1}{2}(e^8-1)$, i) $\frac{\sinh^2 x}{2}\Big]_0^1 = \frac{1}{8}(e^2+e^{-2}-2)$

j) $e^{-1}-e$

E 14) a) $\sqrt{1+x^2}$

b) $\frac{d}{dx}\int_0^{x^2}\sqrt{\sin t+\cos t}\,dt = \frac{d}{dx^2}\int_0^{x^2}\sqrt{\sin t+\cos t}\,dt.\frac{dx^2}{dx}$

$= \sqrt{\sin x^2+\cos x^2}\cdot 2x$

c) $\frac{d}{dx}\int_0^{\sqrt{1-x^2}}(t^3-2t+1)\,dt = \{(\sqrt{1-x^2})^3 - 2\sqrt{1-x^2}+1\}\cdot\frac{d}{dx}\sqrt{1-x^2}$

$= x(1+x^2-\frac{1}{\sqrt{1-x^2}})$

d) $\frac{d}{dx}\int_x^{x^2}\cos t^2 dt = \frac{d}{dx}[\int_0^{x^2}\cos t^2 dt - \int_0^x \cos t^2 dt]$

$= 2x\cos x^4 - \cos x^2$

e) $\frac{d}{dx}(\int_{\sqrt{x}}^{x^2} t\sqrt{1-t^2}\,dt) = \frac{d}{dx}[\int_0^{x^2} t\sqrt{1-t^2}\,dt) - \int_0^{\sqrt{x}} t\sqrt{1-t^2}\,dt]$

$= 2x^3\sqrt{1-x^4} - \frac{1}{2}\sqrt{1-x}$

UNIT 11 METHODS OF INTEGRATION

Structure

11.1 INTRODUCTION

In the last unit we have seen that the definite integral $\int_a^b f(x)\, dx$ represents the signed area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. The Fundamental Theorem of Calculus gives us an easy way of evaluating such an integral, by first finding the antiderivative of the given function, whenever it exists. Starting from this unit, we shall study various methods and techniques of integration. In this unit, we shall consider two main methods: the method of substitution and the method of integration by parts. The next two units will cover some special integrals, which can be evaluated using these two methods.

Objectives

After reading this unit you should be able to

- define the indefinite integral of a function
- evaluate certain standard integrals by finding the antiderivatives of the integrands
- use the rules of the algebra of integrals to evaluate some integrals
- use the method of substitution to simplify and evaluate certain integrals
- integrate by parts a product of two functions.

11.2 BASIC DEFINITIONS

We have seen in Unit 10, that the antiderivative of a function is not unique. More precisely, we have seen that if a function F is an antiderivative of a function f, then F+c is also an antiderivative of f, where c is any arbitrary constant. Now we shall introduce a notation here: We shall use the symbol $\int f(x)\,dx$ to denote the class of all antiderivatives of f. We call it the **indefinite integral** or just the **integral of f.** You must have noticed that we use the same sign $\int$, here that we have used for definite integrals in Unit 10. Thus, if $F(x)$ is an antiderivative of $f(x)$, then we can write

$\int f(x)\,dx = F(x) + c.$

This c is called the constant of integration. As in the case of definite integrals, f(x) is called the integrand and dx indicates that f(x) is integrated with respect to the variable x. For example, In the equation

$$\int (av+b)^4\,dv = \frac{(av+b)^5}{5a} + c,$$

$(av+b)^4$ is the integrand, v is the variable of integration, and $\frac{(av+b)^5}{5a} + c$ is the integral of the integrand $(av+b)^4$.

You will also agree that the indefinite integral of cosx is $\sin x + c$, since we know that sinx is an antiderivative of cosx. Similarly, the indefinite integral of e^{2x} is

$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$, and the indefinite integral of $x^3 + 1$ is $\int (x^3+1)dx = \frac{x^4}{4} + x + c$;

You have seen in Unit 10 that the definite integral $\int_a^b f(x)\,dx$ is a uniquely defined real number whose value depends on a, b and the function f.

On the other hand, the indefinite integral $\int f(x)\,dx$ is a class of functions which differ from one another by constants. It is not a definite number; it is not even a definite function. We say that the indefinite integral is unique upto an arbitrary constant.

Unlike the definite integral which depends on a, b and f, the indefinite integral depends only on f.

All the symbols in the notation $\int_a^b f(x)\,dx$ for the definite integral have an interpretation.

The symbol $\int$ reminds us of summation, a and b give the limits for x for the summation. f(x) dx shows that we are not considering the sum of just the function values, rather we are considering the sum of function values multiplied by small increments in the value of x.

In the case of an indefinite integral, however, the notation $\int f(x)\,dx$ has no similar interpretation. The inspiration for this notation comes from the Fundamental Theorem of Calculus.

Thus, having defined an indefinite integral, let us get acquainted with the various techniques for evaluating integrals.

11.2.1 Standard Integrals

Integration would be a fairly simple matter if we had a list of integral formulas, or a **table of integrals**, in which we could locate any integral that we ever needed to evaluate. But the diversity of integrals that we encounter in practice, makes it impossible to have such a table. One way to overcome this problem is to have a short table of integrals of elementary functions, and learn the techniques by which the range of applicability of this short table can be extended. Accordingly, we build up a table (Table 1) of standard types of integral formulas by inverting formulas for derivatives, which you have already studied in Block 1. Check the validity of each entry in Table 1, by verifying that the derivative of any integral is the given corresponding function.

Table 1

S.No.	Function	Integral
1.	x^n	$\frac{x^{n+1}}{n+1} + c, n \neq -1$
2.	$\sin x$	$-\cos x + c$
3.	$\cos x$	$\sin x + c$
4.	$\sec^2 x$	$\tan x + c$
5.	$\mathrm{cosec}^2 x$	$-\cot x + c$
6.	$\sec x \tan x$	$\sec x + c$
7.	$\mathrm{cosec}\, x \cot x$	$-\mathrm{cosec}\, x + c$
8.	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$, or $-\cos^{-1} x + c$
9.	$\frac{1}{1+x^2}$	$\tan^{-1} x + c$ or $-\cot^{-1} x + c$
10.	$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + c$ or $-\cot^{-1} x + c$
11.	$\frac{1}{x}$	$\ln \lvert x \rvert + c$
12.	e^x	$e^x + c$
13.	a^x	$\frac{a^x}{\ln a} + c$
14.	$\sinh x$	$\cosh x + c$
15.	$\cosh x$	$\sinh x + c$
16.	$\mathrm{sech}^2 x$	$\tanh x + c$
17.	$\mathrm{cosech}^2 x$	$-\coth x + c$
18.	$\mathrm{sech}\, x \tanh x$	$-\mathrm{sech}\, x + c$
19.	$\mathrm{cosech}\, x \coth x$	$-\mathrm{cosech}\, x + c$

Now let us see how to evaluate some functions which are linear combinations of the functions listed in Table 1.

11.2.2 Algebra of Integrals

You are familiar with the rule for differentiation which says

$\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$

There is a similar rule for integration :

Rule 1 $\int [af(x) + bg(x)]dx = a\int f(x)dx + b\int g(x)dx$

This rule follows from the following two theorems.

Theorem 1 If f is an integrable function, then so is kf(x) and

$\int kf(x)dx = k\int f(x)dx$

Proof Let $\int f(x)dx = F(x) + c$,

Then by definition, $\frac{d}{dx}[F(x)+c] = f(x)$

$\therefore \frac{d}{dx}[k\{F(x) + c\}] = kf(x)$

Again, by definition, we have

$\int kf(x)dx = k[F(x) + c] = k\int f(x)dx$

Theorem 2 If f and g are two integrable functions, then f+g is integrable, and we have

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Proof Let $\int f(x)dx = F(x) + c$, $\int g(x)dx = G(x) + c$

Then,

$$\frac{d}{dx}[\{F(x) + c\} + \{G(x) + c\}] = f(x) + g(x)$$

Thus, $\int [f(x) + g(x)]dx = [F(x) + c] + [G(x) + c]$

$$= \int f(x)dx + \int g(x)dx$$

Rule (1) may be extended to include a finite number of functions, that is, we can write

Rule 2 $\int [k_1f_1(x) + k_2f_2(x) + \ldots\ldots\ldots + k_nf_n(x)]\, dx$

$$= k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + \ldots\ldots\ldots\ldots + k_n \int f_n(x)dx$$

We can make use of Rule (2) to evaluate certain integrals which are not listed in Table 1.

Example 1. Let us evaluate $\int (x + \frac{1}{x})^3\, dx$

We know that $(x + \frac{1}{x})^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$. Therefore,

$$\int (x + \frac{1}{x})^3\, dx = \int (x^3 + 3x + \frac{3}{x} + \frac{1}{x^3})\, dx$$

$= x^3dx + 3\int xdx + 3\int \frac{dx}{x} + \int \frac{dx}{x^3}$, ...Rule 2

Using integral formulas 1 and 11 from Table 1, we have

$$\int (x + \frac{1}{x})^3dx = \left(\frac{x^4}{4} + c_1\right) + 3\left(\frac{x^2}{2} + c_2\right) + 3(\ln|x| + c_3) + (\frac{x^{-2}}{-2} + c_4)$$

$$= \frac{x^4}{4} + \frac{3}{2}x^2 + 3\ \ln|x| - \frac{1}{2x^2} + (c_1 + 3c_2 + 3c_3 + c_4)$$

$$= \frac{1}{4}x^4 + \frac{3}{2}x^2 + 3\ \ln|x| - \frac{1}{2x^2} + c$$

Note that $c_1 + 3c_2 + 3c_3 + c_4$ has been replaced by a single arbitrary constant c.

Example 2 Suppose we want to evaluate $\int (2 + 3\sin x + 4e^x)\, dx$

This integral can be written as

$2\int dx + 3\int \sin x\, dx + 4\int e^x\, dx$

$= 2x - 3\cos x + 4e^x + c$

Note that $\int dx = \int 1dx = \int x^0dx = x + c$

Example 3 To evaluate the definite integral $\int_0^1 (x + 2x^2)^2\, dx$, we first find the indefinite integral $\int (x + 2x^2)^2\, dx$.

Thus, $\int (x + 2x^2)^2\, dx = \int (x^2 + 4x^3 + 4x^4)\, dx$

$$= \int x^2dx + 4\int x^3dx + 4\int x^4\, dx$$

$$= \frac{1}{3}x^3 + x^4 + \frac{4}{5}x^5 + c$$

According to our definition of indefinite integral, this gives an antiderivative of $(x+2x^2)^2$ for a given value of c. By using the Fundamental Theorem of Calculus we can now evaluate the definite integral.

$$\int_0^1 (x + 2x^2)^2\, dx = \left(\frac{1}{3}x^3 + x^4 + \frac{4}{5}x^5 + c\right)\Big]_0^1$$

$$= \left(\frac{1}{3} + 1 + \frac{4}{5} + c\right) - c = \frac{32}{15}$$

FTC says that if G (x) is an antiderivative of f(x), then $\int_a^b f(x)\,dx = G(x)]_a^b = G(b) - G(a)$.

Note that for the purpose of evaluating a definite integral, we could take the antiderivative corresponding to c = 0, that is,

$\frac{1}{3}x^3 + x^4 + \frac{4}{5}x^5$, as the constant cancels out.

See if you can do these exercises now.

E

E1) Write down the integrals of the following using Table 1 and Rule 2.

a) (i) x^4 (ii) $x^{-3/2}$ (iii) $4x^{-2}$ (iv) 3

b) (i) $1 - 2x + x^2$ (ii) $(x - \frac{1}{x})^2$ (iii) $(1 + x)^3$

c) (i) $e^x + e^{-x} + 4$ (ii) $4\cos x - 3\sin x + e^x + x$ (iv) $4\operatorname{sech}^2 x + e^x - 8x$

d) (i) $\frac{2}{\sqrt{1-x^2}} + \frac{5}{x}$ (ii) $\frac{2x^2+5}{x^2+1}$

e) (i) $ax^3 + bx^2 + cx + d$ (ii) $(\sqrt{x} - \frac{1}{\sqrt{x}})^2$

f) (i) $\frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x}$ (ii) $(2+x)(3 - \sqrt{x})$

E2) Evaluate the following definite integrals.

a) (i) $\int_3^6 x^4 \, dx$ (ii) $\int_1^2 \frac{1+x}{x^2} \, dx$

b) (i) $\int_2^4 (x + \frac{1}{x})^2 \, dx$ (ii) $\int_0^1 (x+1)^3 \, dx$

You have seen that with the help of Rule 2 we could evaluate a number of integrals. But still there are certain integrals like $\int \sin 2x \, dx$ which cannot be evaluated by using Rule 2. The method of substitution which we are going to describe in the next section will come in handy in these cases.

11.3 INTEGRATION BY SUBSTITUTION

In this section we shall study the first of the main methods of integration dealt with in this unit: the method of substitution. This is one of the most commonly used techniques of integration. We shall illustrate its application through a number of examples.

11.3.1 Method of Substitution

The following theorem will lead us to this method.

Theorem 3 If $\int f(v)dv = F(v) + c$, then on substituting v by g(x), we get $\int f[g(x)]\, g'(x)\, dx = \int f(v)dv$.

Proof We shall make use of the chain rule for derivatives (Unit 3) to prove this theorem. Since $\int f(v)dv = F(v) + c$, we can write $\frac{dF(v)}{dv} = f(v)$. Now if we write v as a function of x, say $v = g(x)$, then

$$\frac{d}{dx} F[g(x)] = \frac{dF[g(x)]}{dg(x)} \cdot \frac{dg(x)}{dx} \text{ by chain rule}$$

$$= f[g(x)] \cdot \frac{dg(x)}{dx} \text{ since } v = g(x)$$

$$= f[g(x)] \cdot g'(x)$$

This shows that F[g(x)] is an antiderivative of f[g(x)g'(x). This means that

$$\int f[g(x)]g'(x)\, dx = F[g(x)] + c = F(v) + c = \int f(v)dv.$$

The statement of this theorem by itself may not seem very useful to you. But it does simplify our task of evaluating integrals. For example, to evaluate $\int \sin 2x\, dx$, we could take $v = g(x) = 2x$ and get

$$\int \sin 2x\, dx = \frac{1}{2}\int \sin 2x\,(2)\, dx$$

$$= \frac{1}{2}\int \sin v\, dv \text{ by Theorem 3, since } g(x) = 2x \text{ and } g'(x) = 2.$$

$$= \frac{-\cos v}{2} + c$$

$$= -\frac{\cos 2x}{2} + c$$

We make a special mention of the following three cases which follow from Theorem 3.

Case i) If $f(v) = v^n$, $n \neq -1$ and $v = g(x)$, then, by Formula I of Table 1,

$$\int [g(x)]^n g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + c$$

Case ii) If $f(v) = 1/v$ and $v = g(x)$.

then, by Formula II of Table 1

$$\int \frac{g'(x)}{g(x)}\, dx = \ln |g(x)| + c$$

Case iii) If $\int f(x)\, dx = F(x) + c$, then

$$\int f[g(x)]\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(v)\, dv, \text{ where } v = g(x)$$ [The limits of integration are g(a) and g(b)

$$= F(v)\Big]_{g(a)}^{g(b)}$$

since $x = a \Rightarrow v = g(x) = g(a)$, and $x = b \Rightarrow v = g(x) = g(b)$.]

We shall be using these three cases very often. Their usefulness is evident from the following examples.

Example 4 Let us integrate $(2x + 1)(x^2 + x + 1)^5$

For this we observe that $\frac{d}{dx}(x^2+x+1) = 2x+1$

Thus, $\int (2x+1)(x^2+x+1)^5 \, dx$ is of the form $\int [g(x)]^n g'(x)\, dx$ and hence can be evaluated as in i) above.

Therefore, $\int (2x+1)(x^2+x+1)^5 \, dx = \frac{1}{6}(x^2+x+1)^6 + c$.

Alternatively, to find $\int (2x+1)(x^2+x+1)^5 \, dx$, we can substitute x^2+x+1 by u. This means

$$\frac{du}{dx} = 2x + 1$$

Therefore, $\int (2x+1)(x^2+x+1)^5 \, dx = \int u^5 \, du$ by Theorem 3

$= \frac{1}{6} u^6 + c$ by Formula 1 from Table 1.

$= \frac{1}{6}(x^2+x+1)^6 + c$

Example 5 Let us evaluate $\int (ax + b)^n \, dx$.

$$\int (ax+b)^n dx = \frac{1}{a}\int a(ax+b)^n \, dx$$

$$= \frac{1}{a}\int (ax+b)^n \frac{d}{dx}(ax+b)\, dx$$

Therefore, when $n \neq -1$,

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,$$

and when $n = -1$,

$$\int (ax+b)^n dx = \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c$$

Example 6 Suppose we want to evaluate the definite integral

$$\int_0^2 \frac{x+1}{x^2+2x+3}\, dx$$

We put $x^2+2x+3 = u$. This implies $\frac{du}{dx} = 2(x+1)$. Further, when $x = 0$, $u = 3$, and when $x = 2$, $u = 11$. Thus,

$$\int_0^2 \frac{x+1}{x^2+2x+1}\, dx = \frac{1}{2}\int_0^2 \frac{1}{u}\frac{du}{dx}\, dx = \frac{1}{2}\int_3^{11}\frac{du}{u} = \frac{1}{2}\ln|u|\Big]_3^{11}$$

$$= \frac{1}{2}(\ln 11 - \ln 3) = \frac{1}{2}\ln\frac{11}{3}$$

Example 7 To evaluate $\int xe^{2x^2}dx$, we substitute $2x^2 = u$. Since $\frac{du}{dx} = 4x$, we can write,

$$\int xe^{2x^2}dx = \frac{1}{4}\int e^{2x^2} 4x\, dx = \frac{1}{4}\int e^u \frac{du}{dx}\, dx$$

$$= \frac{1}{4}\int e^u du = \frac{1}{4}e^u + c.$$

$$= \frac{1}{4}e^{2x^2} + c$$

On the basis of the rules discussed in this section, you will be able to solve this exercise.

E3) Evaluate

a) $\int \sqrt{5x-3}\, dx$ b) $\int (2x+1)^6\, dx$ c) $\int_1^3 \frac{dx}{4+5x}$

d) $\int \frac{5\, dx}{10x+7}$ e) $\int \frac{x+1}{x^2+2x+7}\, dx$ f) $\int_2^3 \frac{3x^2+2x+1}{x^3+x^2+x-8}\, dx$

g) $\int x^{1/3} \sqrt{x^{4/3}-1}\, dx$ h) $\int \frac{x\, dx}{1-3x^2}$

Now we shall use the method of substitution to integrate some trigonometric functions. Let's start with sin ax.

Example 8 To evaluate $\int \sin ax\,dx$, we proceed in the same manner as we did for $\int \sin 2x\,dx$. We make the substitution $ax = u$.

This gives $\frac{du}{dx} = a$. Thus,

$$\int \sin ax\,dx = \frac{1}{a}\int \sin u \cdot \frac{du}{dx} \cdot dx$$

$$= \frac{1}{a}\int \sin u\,du = \frac{-\cos u}{a} + c$$

$$= -\frac{1}{a}\cos ax + c$$

E4) Proceeding exactly as in Example 8, fill up the blanks in the table below.

S.No.	f(x)	$\int f(x)\,dx$
1.	sinax	$-\frac{1}{a}\cos ax + c$
2.	cosax	$\frac{1}{a}\sin ax + c$
3.	$\sec^2 ax$	....
4.	$\operatorname{cosec}^2 ax$	....
5.	secax tanax	....
6.	cosecax cotax	....
7.	e^{ax}	....
8.	a^{mx}	....

Example 9 Suppose we want to evaluate

i) $\int \cot x\,dx$; ii) $\int \tan x\,dx$ and iii) $\int \operatorname{cosec} 2x\,dx$

i) We can write

$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$. Now, since $\frac{d}{dx}\sin x = \cos x$, this integral falls in the category of case ii) mentioned earlier, and thus, $\int \cot x\,dx = \ln|\sin x| + c$

ii) To evaluate $\int \tan x\,dx$, we write

$$\int \tan x\,dx = \int \frac{\sec x\,\tan x}{\sec x}\,dx$$

$$= \ln|\sec x| + c, \text{ as } \frac{d}{dx}\sec x = \sec x\tan x$$

iii) To integrate cosec2x we write

$$\int \operatorname{cosec} 2x\,dx = \frac{1}{2}\int \frac{2\operatorname{cosec} 2x(\operatorname{cosec} 2x - \cot 2x)}{\operatorname{cosec} 2x - \cot 2x}\,dx$$

Here again, $\frac{d}{dx}(\operatorname{cosec} 2x - \cot 2x) = 2\operatorname{cosec} 2x(\operatorname{cosec} 2x - \cot 2x)$

This means $\int \operatorname{cosec} 2x\,dx = \frac{1}{2}\ln|\operatorname{cosec} 2x - \cot 2x| + c$

In this example we have used some 'tricks' to put the integrand in some standard form. After you study various examples and try out a number of exercises, you will be able to decide on the particular substitution or the particular trick which will reduce the given integrand to one of the known forms. Let's look at the next example now.

Example 10 Let us evaluate $\int e^{\sin^2 x} \sin 2x \, dx$

If we put $\sin^2 x = u$, then $\frac{du}{dx} = 2\sin x \cos x = \sin 2x$

Therefore, $\int e^{\sin^2 x} \sin 2x \, dx = \int e^u \, du$

$= e^u + c = e^{\sin^2 x} + c$

See if you can solve this exercise now.

E5) Evaluate the following integrals

a) $\int \sec x \, dx$ b) $\int_0^{\pi/2} \sin^2 x \cos x \, dx$ c) $\int e^{\tan x} \sec^2 x \, dx$

11.3.2 Integrals using Trigonometric Formulas

In this section, we shall evaluate integrals with the help of the following trigonometric formulas:

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$

$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$

$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$

In each of these formulas you will find that on the left hand side we have either a power of a trigonometric function or a product of two trigonometric functions. And on the right hand side we have a sum (or difference) of two trigonometric functions. You will realise that the functions on the right hand side can be easily integrated by making suitable substitutions.

The following examples will illustrate how we make use of the above formulas in evaluating certain integrals.

Example 11 To evaluate $\int \cos^3 ax\, dx$. We write

$$\int \cos^3 ax\, dx = \int \left(\frac{3}{4}\cos ax + \frac{1}{4}\cos 3ax\right) dx$$

$$= \frac{3}{4}\int \cos ax\, dx + \frac{1}{4}\int \cos 3ax\, dx$$

$$= \frac{3}{4a}\sin ax + \frac{1}{12a}\sin 3ax + c \qquad \text{(see E 4)}$$

Example 12 Let us evaluate i) $\int \sin 3x \cos 4x\, dx$ and ii) $\int \sin x \sin 2x \sin 3x\, dx$

Here the integrand is in the form of a product of trigonometric functions. We shall write it as a sum of trigonometric functions so that it can be integrated easily.

i) $$\int \sin 3x \cos 4x\, dx = \int \frac{1}{2}(\sin 7x - \sin x)\, dx$$

$$= \frac{1}{2}\int \sin 7x\, dx - \frac{1}{2}\int \sin x\, dx$$

$$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + c$$

ii) To evaluate $\int \sin x \sin 2x \sin 3x\, dx$, again we express the product $\sin x \sin 2x \sin 3x$ as a sum of trigonometric functions.

$$\sin x \sin 2x \sin 3x = \frac{1}{2}\sin x(\cos x - \cos 5x)$$

$$= \frac{1}{2}\sin x \cos x - \frac{1}{2}\sin x \cos 5x$$

$$= \frac{1}{4}\sin 2x - \frac{1}{4}(\sin 6x - \sin 4x)$$

Therefore, $\int \sin x \sin 2x \sin 3x \, dx$

$$= \frac{1}{4}\int \sin 2x \, dx + \frac{1}{4}\int \sin 4x \, dx - \frac{1}{4}\int \sin 6x \, dx$$

$$= -\frac{1}{8}\cos 2x - \frac{1}{16}\cos 4x + \frac{1}{24}\cos 6x + c$$

Try to do some exercises now. You will be able to solve them either by applying the trigonometric formulas mentioned in the beginning of this section or by using the method of substitution. Don't be scared by the number of integrals to be evaluated. The more integrals you evaluate, the more skilled you will become. You have to practise a lot to be able to decide on the best method to be applied for evaluating any given integral.

E E6) Evaluate each of the following integrals.

a) i) $\int \sin^5 x \cos x \, dx$ ii) $\int \frac{\cos x}{\sin^3 x} dx$ iii) $\int_{\pi/6}^{\pi/3} \cot 2x \operatorname{cosec}^2 2x \, dx$

iv) $\int \sin 2\theta \, e^{\cos 2\theta} d\theta$ v) $\int_0^{\pi/2} \sin\theta(1+\cos^4\theta) \, d\theta$

b) i) $\int (1+\cos\theta)^4 \sin\theta \, d\theta$ ii) $\int_0^{\pi/3} \frac{\sec^2\theta d\theta}{(1-5\tan\theta)^3}$

iii) $\int_0^{\pi/4} \sec\theta \tan\theta(1+\sec\theta)^3 \, d\theta$

c) i) $\int \sin^4\theta \, d\theta$ ii) $\int \sin 3\theta \cos\theta \, d\theta$

iii) $\int_0^{\pi/2} \cos 5\theta \cos\theta \, d\theta$ iv) $\int_0^{\pi/2} \cos\theta \cos 2\theta \cos 4\theta \, d\theta$

E7) The cost of a transistor radio is Rs. 700/-. Its value is depreciating with time according to the formula $\frac{dv}{dt} = \frac{-500}{(1+t^2)}$, where Rs. v is its value t years after its purchase. What will be its value 3 years after its purchase? (Don't forget the constant of integration! Think how you can find it with the help of the given information.)

11.3.3 Trigonometric and Hyperbolic Substitution

Various trigonometric and hyperbolic identities like $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, $\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$ and so on, prove very useful while evaluating certain integrals. In this section we shall see how.

A trigonometric or hyperbolic substitution is generally used to integrate expressions involving $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or a^2+x^2. We suggest the following substitutions.

Expression Involved	Substitution
$\sqrt{a^2-x^2}$	$x = a\sin\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$ or $a\sinh\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $a\cosh\theta$
a^2+x^2	$x = a\tan\theta$

Thus, to evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$, put $x = a\sin\theta$. Then we know that

$\frac{dx}{d\theta} = a\cos\theta$. This means we can write

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta + c$$

$$= \sin^{-1}(x/a) + c$$

Similarly to evaluate $\int \frac{dx}{a^2+x^2}$, we shall put $x = a\tan\theta$

Since $\frac{dx}{d\theta} = a\sec^2\theta\, d\theta$, we get

$$\int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta d\theta}{a^2+a^2\tan^2\theta}$$

$$= \int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{\theta}{a} + c$$

$$= \frac{1}{a}\tan^{-1}(x/a) + c$$

We can also evaluate $\int \frac{dx}{\sqrt{a^2+x^2}}$, by substituting $x = a\tan\theta$

This gives $\frac{dx}{d\theta} = a\sec^2\theta$

$$\text{Thus, } \int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2+a^2\tan^2\theta}} = \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right| + c$$

We can also evaluate this integral by putting $x = a\sinh\theta$. With this substitution we get,

$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}(x/a) + c$, and we know that

$\sinh^{-1}(x/a) = \ln. \frac{x+\sqrt{x^2+a^2}}{a}$ (see Unit 5)

Similarly,

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) + c,$$

$$= \ln. \frac{x+\sqrt{x^2-a^2}}{a} + c$$

and $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}(x/a) + c$

Let us put these results in the form of a table.

Table 3

S.No.	f(x)	$\int f(x)\,dx$
1.	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}(x/a)+c$
2.	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}(x/a)+c$
3.	$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}(x/a)+c$
4.	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2+a^2}}{a}\right\rvert+c$ or $\sinh^{-1}(x/a)+c$
5.	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert+c$ or $\cosh^{-1}(x/a)+c$

Sometimes the integrand does not seem to fall in any of the types mentioned in Table 3, but it is possible to modify or rearrange it so that it conforms to one of these types. We shall illustrate this through some examples.

Example 13 Suppose we want to evaluate $\int_1^2 \frac{dx}{\sqrt{2x-x^2}}$

Let us try to rearrange the terms in the integrand $\frac{1}{\sqrt{2x-x^2}}$ to suit us. You will see that

$$\int_1^2 \frac{dx}{\sqrt{2x-x^2}} = \int_1^2 \frac{dx}{\sqrt{1-(x-1)^2}}$$

If we put $x-1=v$, $\frac{dv}{dx}=1$ and

$$\int_1^2 \frac{dx}{\sqrt{2x-x^2}} = \int_0^1 \frac{dv}{\sqrt{1-v^2}}.$$ Note the new limits of integration.

This integral is finally in the form that we want and using the first formula in Table 3 we get

$$\int_1^2 \frac{dx}{\sqrt{2x-x^2}} = \sin^{-1}v\Big]_{v=0}^{1}$$

$$= \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Example 14 The integrand in $\int \frac{x^2}{1+x^6}\,dx$ does not again fall into the types mentioned in Table 3. But let's see what we can do.

If we put $x^3 = u$, $\frac{du}{dx} = 3x^2$. Thus,

$x=0 \Rightarrow u = x^3 = 0$ and $x = 1 \Rightarrow u = 1$.

$$\int_0^1 \frac{x^2}{1+x^6}\,dx = \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^6}\,dx$$

$$= \frac{1}{3}\int_0^1 \frac{1}{1+u^2}\,\frac{du}{dx}\,dx$$

$$= \frac{1}{3}\int_0^1 \frac{1}{1+u^2}\,du,$$ by Theorem 3

Here the integrand $\frac{1}{1+u^2}$ can be evaluated using formula 2 in Table 3.

Thus, we get

$$\frac{1}{3}\int_0^1 \frac{3x^2}{1+x^6}\,dx = \frac{1}{3}\tan^{-1} u \Big]_0^1$$

$$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}$$

If you have followed this discussion, you will certainly be able to solve this exercise.

E8) Integrate each of the following with respect to the corresponding variables.

i) $\frac{1}{\sqrt{9-x^2}}$ ii) $\frac{1}{\sqrt{u^2-4}}$ iii) $\frac{1}{1+4x^2}$ iv) $\frac{1}{2x^2+5}$

v) $\frac{x}{\sqrt{x^4-1}}$ vi) $\frac{t^2}{t^6+16}$ vii) $\frac{u^2}{\sqrt{4-u^6}}$

viii) $\frac{1}{\sqrt{2x-x^2}}$ ix) $\frac{1}{\sqrt{1+x+x^2}}$ x) $\frac{1}{\sqrt{y^2+6y+5}}$

xi) $\frac{x^2}{1+x^2}$ (Hint: $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$)

11.3.4 Two Properties of Definite Integrals

We have already derived some properties of the definite integrals in Unit 10. These are the

i) Constant Function Property : $\int_a^b c\,dx = c(b-a)$

ii) Constant Multiple Property : $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

iii) Interval Union Property : $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$

iv) Comparison Property : If $c \le f(x) \le d \ \forall\, x \in [a,b]$,

then, $c(b-a) \le \int_a^b f(x)\,dx \le d(b-a)$.

Now we shall use the method of substitution to derive two more properties to add to this list. Let's consider them one by one.

v) $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$ for any integrable function f.

We already know that

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^a f(x)\,dx.$$

Now if we put $x = a-y$ in the second integral on the right hand side, then since $\frac{dy}{dx} = -1$, we get

$$\int_{a/2}^{a} f(x)\,dx = -\int_{a/2}^{0} f(a-y)\,dy = \int_0^{a/2} f(a-y)\,dy = \int_0^{a/2} f(a-x)\,dx,$$

since x is a dummy variable.

Thus $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$.

The usefulness of this property will be clear to you from the following example.

Example 15 Let us evaluate i) $\int_0^{\pi} \sin^4x \cos^5x\,dx$ and ii) $\int_0^{2\pi} \cos^3x\,dx$

i) Using property v), we can write

$$\int_0^{\pi} \sin^4x\cos^5x\,dx = \int_0^{\pi/2} \sin^4x\cos^5x\,dx + \int_0^{\pi/2} \sin^4(\pi-x)\cos^5(\pi-x)\,dx$$

$$= \int_0^{\pi/2} \sin^4x\cos^5x\,dx + \int_0^{\pi/2} \sin^4x(-\cos x)^5dx$$

$\sin(\pi-x) = \sin x$, and $\cos(\pi-x) = -\cos x$

$$= \int_0^{\pi/2} \sin^4x\cos^5x\,dx - \int_0^{\pi/2} \sin^4x\cos^5x\,dx$$

$$= 0$$

ii) $$\int_0^{2\pi} \cos^3x\,dx = \int_0^{\pi} \cos^3x\,dx + \int_0^{\pi} \cos^3(2\pi-x)\,dx$$

$$= \int_0^{\pi} \cos^3x\,dx + \int_0^{\pi} \cos^3x\,dx$$

$\cos(2\pi-x) = \cos x$

$$= 2\int_0^{\pi} \cos^3x\,dx$$

$$= 2\left[\int_0^{\pi/2} \cos^3x\,dx + \int_0^{\pi/2} \cos^3(\pi-x)\,dx\right]$$

$$= 2\left[\int_0^{\pi/2} \cos^3x\,dx - \int_0^{\pi/2} \cos^3x\,dx\right] = 0$$

Our next property greatly simplifies some integrals when the integrands are even or odd functions.

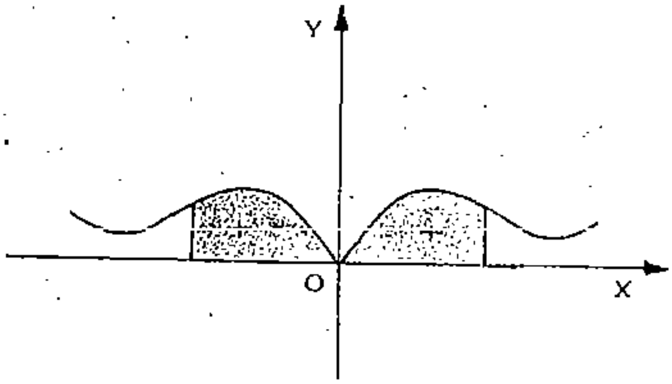
vi) If f is an even function of x, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

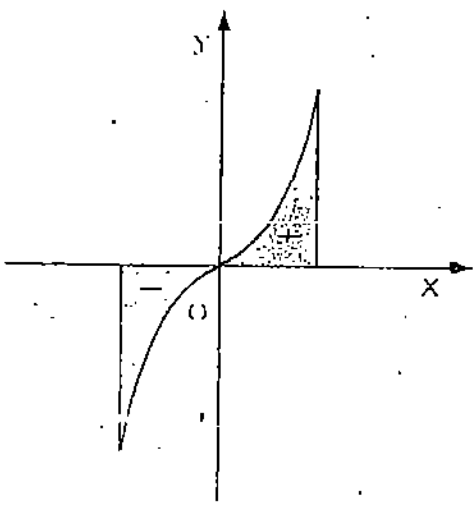
and if f is an odd function, then,

$$\int_{-a}^{a} f(x)\,dx = 0$$

This is also obvious from Fig. 1(a) and (b)



(a)



(b)

Fig. 1

We shall prove the result for even functions. The result for odd functions follows easily and is left to you as an exercise. (see E 9) c)).

So let f be an even function of x in $[-a,a]$, that is, $f(-x) = f(x) \forall x \in [-a,a]$.
Then,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

If we put $x = -y$ in the first integral on the right hand side, we get

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-y)\,dy = \int_{0}^{a} f(y)\,dy = \int_{0}^{a} f(x)\,dx.$$

Thus, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Using this property we can directly say that

$$\int_{-\pi/2}^{\pi/2} \sin x\,dx = 0,\ \int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_{0}^{\pi/2} \cos x\,dx = 2\sin x\Big]_{0}^{\pi/2} = 2.$$

Try to solve this exercise now.

E E9) a) Evaluate

$$\int_{0}^{\pi} \sin^5 x \cos^3 x\,dx$$

b) Show that $\int_{0}^{\pi/2} \sin 2x \ln(\tan x)\,dx = 0$

c) Prove that $\int_{-a}^{a} f(x)\,dx = 0$ if f is an odd function of x.

In this section we have seen how the method of substitution enables us to substantially increase our list of integrable functions. (Here by "integrable function" we mean a function which *we* can integrate!) We shall discuss another technique in the next section.

11.4 INTEGRATION BY PARTS

In this section we shall evolve a method for evaluating integrals of the type $\int u(x)v(x)\,dx$, in which the integrand $u(x)v(x)$ is the product of two functions. In other words, we shall first evolve the integral analogue of

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

and then use that result to evaluate some standard integrals.

11.4.1 Integrals of a Product of Two Functions

We can calculate the derivative of the product of two functions by the formula

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

Let us rewrite this as

$$u(x)\frac{d}{dx}v(x) = \frac{d}{dx}[u(x)v(x)] - v(x)\frac{d}{dx}u(x)$$

Integrating both the sides with respect to x, we have

$$\int u(x)\frac{d}{dx}(v(x))\,dx = \int \frac{d}{dx}(u(x)\,v(x))\,dx \int v(x)\frac{d}{dx}(u(x))\,dx \text{ , or}$$

$$\int u(x)\frac{d}{dx}(v(x))\,dx = u(x)v(x) - \int v(x)\frac{d}{dx}(u(x))\,dx \qquad \text{.......... (1)}$$

To express this in a more symmetrical form, we replace $u(x)$ by $f(x)$, and put

$\frac{d}{dx}v(x) = g(x)$. This means $v(x) = \int g(x)\,dx$.

As a result of this substitution, (1) takes the form

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int \{f'(x)\int g(x)\,dx\}\,dx$$

This formula may be read as :

The integral of the product of two functions = First factor × integral of second factor − integral of (derivative of first factor × integral of second factor)

It is called the **formula for integration by parts.** This formula may appear a little complicated to you. But the success of this method depends upon choosing the *first* factor in such a way that the second term on the right-hand side may be easy to evaluate. It is also essential to choose the second factor such that it can be easily integrated.

The following examples will show you the wide variety of integrals which can be evaluated by this technique. You should carefully study our choice of first and second functions in each example. You may also try to evaluate the integrals by reversing the order of functions. This will make you realise why we have chosen these functions the way we have.

Example 16 Let us use the method of integration by parts to evaluate $\int xe^x\,dx$.

In the integrand xe^x we choose x as the first factor and e^x as the second factor. Thus, we get

$$\int xe^x dx = x\int e^x dx - \int \{\frac{d}{dx}(x)\int e^x dx\}\,dx$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

Example 17 To evaluate $\int_0^{\pi/2} x^2 \cos x\,dx$, We shall take x^2 as the first factor and $\cos x$ as the second. Let us first evaluate the corresponding indefinite integral.

$$\int x^2 \cos x\,dx = x^2 \int \cos x\,dx - \int \{ \frac{d}{dx}(x^2) \int \cos x dx \}\,dx$$
$$= x^2 \sin x - \int 2x\sin x\,dx$$
$$= x^2 \sin x - 2\int x\sin x\,dx$$

We shall again use the formula of integration by parts to evaluate $\int x\sin x\,dx$. Thus,

$$\int x\sin x\,dx = x(-\cos x) - \int (1)(-\cos x)\,dx \quad (f(x) = x,\ g(x) = \sin x)$$
$$= -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + c$$

Hence,

$$\int x^2 \cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + c$$

Note that we have written the arbitrary constant as c instead of 2c.

$$\text{Now} \int_0^{\pi/2} x^2 \cos x\,dx = (x^2\sin x + 2x\cos x - 2\sin x + c)\Big|_0^{\pi/2}$$
$$= \frac{\pi^2}{4} - 2$$

Example 18 Let us now evaluate $\int x \ln|x|\,dx$

Here we take $\ln|x|$ as the first factor since it can be differentiated easily, but cannot be integrated that easily. We shall take x to be the second factor.

$$\int x \ln|x|\,dx = \int (\ln|x|)x\,dx$$
$$= (\ln|x|)\frac{x^2}{2} - \int (\frac{1}{x})(\frac{x^2}{2})\,dx$$
$$= \frac{1}{2}x^2 \ln|x| - \frac{1}{2}\int x\,dx$$
$$= \frac{1}{2}x^2 \ln|x| - \frac{1}{4}x^2 + c$$

While choosing $\ln|x|$ as the first factor, we mentioned that it cannot be integrated easily. The method of integration by parts, in fact, helps us in integrating $\ln x$ too.

Example 19 We can find $\int \ln x\,dx$ by taking $\ln x$ as the first factor and 1 as the second factor. Thus,

$$\int \ln x\,dx = \int (\ln x)(1)\,dx$$
$$= \ln x \int 1 dx - \int (\frac{1}{x} \int 1\,dx)\,dx$$
$$= (\ln x)(x) - \int \frac{1}{x}(x)\,dx$$
$$= x\ln x - \int dx = x\ln x - x + c$$
$$= x\ln x - x\ln e + c \quad \text{since } \ln e = 1$$
$$= x\ln(x/e) + c$$

The trick used in Example 19, that is, considering 1 (unity) as the second factor, helps us to evaluate many integrals which could not be evaluated earlier.

You will be able to solve the following exercises by using the method of integration by parts.

E10) Evaluate

a) $\int x^2 \ln x \, dx$ Take $f(x) = \ln x$ and $g(x) = x^2$

b) $\int (1+x)e^x \, dx$ Take $f(x) = 1+x$ and $g(x) = e^x$

c) $\int (1+x^2)e^x \, dx$

d) $\int x^2 \sin x \cos x \, dx$ Take $f(x) = x^2$ and $g(x) = \sin x \cos x$
$= \frac{1}{2} \sin 2x.$

E11) Evaluate the following integrals by choosing 1 as the second factor.

a) $\int \sin^{-1} x \, dx$ b) $\int_0^1 \tan^{-1} x \, dx$ c) $\int \cot^{-1} x \, dx$

E12) Integrate

a) $x\sin^{-1}x$ b) $\ln(1+x^2)$ w.r.t. x.

11.4.2 Evaluation of $\int e^{ax}\sin bx\, dx$ and $\int e^{ax}\cos bx\, dx$

To evaluate $\int e^{ax} \sin bx\, dx$ and $\int e^{ax} \cos bx\, dx$, we use the formula for integration by parts.

$$\int e^{ax}\sin bx\, dx = (e^{ax})\left(-\frac{1}{b}\cos bx\right) - \int (ae^{ax})\left(-\frac{1}{b}\cos bx\right) dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\, dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[(e^{ax})\left(\frac{1}{b}\sin bx\right) - \int \left(e^{ax}\frac{a}{b}\sin bx\right) dx\right]$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\, dx$$

Therefore, you will notice that the last integral on the right hand side is the same as the integral on the left hand side. Now we transfer the third term on the right to the left hand side, and obtain,

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\sin bx\, dx = e^{ax}\left(\frac{a}{b^2}\sin bx - \frac{1}{b}\cos bx\right)$$

This means,

$$\int e^{ax}\sin bx\, dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx) + c$$

We can similarly show that

$$\int e^{ax}\cos bx\, dx = \frac{1}{a^2+b^2}e^{ax}(a\cos bx + b\sin bx) + c$$

If we put $a = r\cos\theta$, $b = r\sin\theta$, these formulas become

$$\int e^{ax}\sin bx\, dx = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin(bx-\theta) + c$$

$$\int e^{ax}\cos bx\, dx = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\cos(bx-\theta) + c, \text{ where } \theta = \tan^{-1}\frac{b}{a}.$$

Example 20 Using the formulas discussed in this sub-section, we can easily check that

i) $\int e^x \sin x\, dx = \frac{1}{\sqrt{2}}e^x\sin\left(x - \frac{\pi}{4}\right) + c$

and

ii) $\int e^x \cos\sqrt{3}x\, dx = \frac{1}{2}e^x\cos\left(\sqrt{3}x - \frac{\pi}{3}\right) + c$

Example 21 To evaluate $\int e^{2x} \sin x \cos 2x\, dx$, we shall first write

$\sin x \cos 2x = \frac{1}{2}(\sin 3x - \sin x)$ as in Sec. 3.

Therefore,

$$\int e^{2x}\sin x\cos 2x\, dx = \frac{1}{2}\int e^{2x}\sin 3x\, dx - \frac{1}{2}\int e^{2x}\sin x\, dx$$

Now the two integrals on the right hand side can be evaluated. We see that

$$\int e^{2x}\sin 3x\, dx = \frac{1}{\sqrt{13}}e^{2x}\sin\left(3x - \tan^{-1}\frac{3}{2}\right) + c$$

and

$$\int e^{2x}\sin x\, dx = \frac{1}{\sqrt{5}}e^{2x}\sin\left(x - \tan^{-1}\frac{1}{2}\right) + c'$$

Hence

$$\int e^{2x}\sin x\cos 2x\, dx = e^{2x}\left[\frac{1}{\sqrt{13}}\sin\left(3x - \tan^{-1}\frac{3}{2}\right) - \quad \sin\left(x - \tan^{-1}\frac{1}{2}\right)\right] + c$$

Example 22 Suppose we want to evaluate $\int x^3 \sin(a\ln x)\,dx$.

Let $\ln x = u$. This implies $x = e^u$ and $du/dx = 1/x$.

Then, $\int x^3 \sin(a\ln x)\,dx = \int x^4 \sin(a\ln x)(1/x)\,dx$

$$= \int e^{4u} \sin au\,du$$

$$= \frac{1}{\sqrt{16+a^2}}\, e^{4u} \sin(au - \tan^{-1}(a/4)) + c$$

$$= \frac{1}{\sqrt{16+a^2}}\, x^4 \sin\left(a\ln x - \tan^{-1}\frac{a}{4}\right) + c$$

Why don't you try some exercises now.

E

E13) Evaluate the following integrals

a) $\int e^{2x}\cos 4x\,dx$ b) $\int e^{3x}\sin 3x\,dx$ c) $\int e^{4x}\cos x.\cos 2x\,dx$

d) $\int e^{2x}\cos^2 x\,dx$ e) $\int \cosh ax \sin bx\,dx$ (write cosh ax in terms of the exponential function)

f) $\int xe^{ax}\sin bx\,dx$

11.4.3 Evaluation of $\int \sqrt{a^2-x^2}\,dx$, $\int \sqrt{a^2+x^2}\,dx$, and $\int \sqrt{x^2-a^2}\,dx$

In this sub-section, we shall see that integrals like $\int \sqrt{a^2-x^2}\,dx$, $\int \sqrt{a^2+x^2}\,dx$ and $\int \sqrt{x^2-a^2}\,dx$ can also be evaluated with the help of the formula for integration by parts and Table 3.

$$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-x^2}\,(1)\,dx$$

$$= \sqrt{a^2-x^2} \times x - \int \left(\frac{-x}{\sqrt{a^2-x^2}} \times x\right) dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int \sqrt{a^2-x^2}\,dx$$

Shifting the last term on the right hand side to the left we get,

$$2\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

Using the formula,

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$, we obtain

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

Similarly, we shall have,

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + c$$

$$= \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\frac{x+\sqrt{a^2+x^2}}{a} + c$$

and

$$\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$$

$$= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\frac{x+\sqrt{x^2-a^2}}{a} + c$$

Example 23 Let us evaluate $\int_0^1 \sqrt{x+x^2}\,dx$

Now $\int_0^1 \sqrt{x+x^2}\,dx = \int_0^1 \sqrt{(x+1/2)^2 - 1/4}\,dx$

Let $x + \frac{1}{2} = u$. Then,

$$\int_0^1 \sqrt{x+x^2}\,dx = \int_{1/2}^{3/2} \sqrt{u^2-1/4}\,du$$

$$= \left\{\frac{1}{2}u\sqrt{u^2-1/4} - \frac{1}{8}\ln\frac{u+\sqrt{u^2-1/4}}{1/2}\right\}\Bigg]_{1/2}^{3/2}$$

$$= \frac{3\sqrt{2}}{4} - \frac{1}{8}\ln(3+2\sqrt{2})$$

Surely, you will be able to do these exercises now.

E14) Verify that

a) $\int \sqrt{a^2+x^2}\, dx = \frac{1}{2} x \sqrt{a^2+x^2} + \frac{a^2}{2} \ln \frac{x + \sqrt{a^2+x^2}}{a} + c$

b) $\int \sqrt{x^2-a^2}\, dx = \frac{1}{2} x \sqrt{x^2-a^2} - \frac{a^2}{2} \ln \frac{x + \sqrt{x^2-a^2}}{a} + c$

In the next sub-section we shall consider another type of integrand which occurs quite frequently in mathematics.

11.4.4 Integrals of the Type $\int e^x [f(x) + f'(x)]\, dx$

We first prove the formula

$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$ and see how it can be used in integrating some functions.

By the formula for integration by parts

$$\int e^x f(x)\, dx = \int f(x)\, e^x\, dx$$
$$= f(x)\, e^x - \int f'(x)\, e^x\, dx + c$$

This implies

$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$$

Example 24 Let us evaluate the following integrals.

i) $\int \frac{1+x}{(2+x)^2} e^x \, dx$ ii) $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} \, dx$

We take up (i) first,

i) $\int \frac{1+x}{(2+x)^2} e^x \, dx = \int \frac{(2+x)-1}{(2+x)^2} e^x \, dx$

$= \int \left[\frac{1}{2+x} + \frac{-1}{(2+x)^2} \right] e^x \, dx$

$= \frac{1}{2+x} e^x + c$, since $\frac{-1}{(2+x)^2} = \frac{d}{dx}\left(\frac{1}{2+x}\right)$

Now we shall evaluate ii)

ii) $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} \, dx$

$= \int \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{2\cos^2\frac{x}{2}} e^{-x/2} \, dx$

$= \frac{1}{2}\int \sec\frac{x}{2} e^{-x/2} dx - \frac{1}{2}\int \tan\frac{x}{2} \sec\frac{x}{2} e^{-x/2} \, dx$

Now

$\int \sec\frac{x}{2} e^{-x/2} dx = (\sec\frac{x}{2})(-2e^{-x/2}) - \int (\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2})(-2e^{-x/2}) \, dx$

$= -2\sec\frac{x}{2} e^{-x/2} + \int \sec\frac{x}{2}\tan\frac{x}{2} e^{-x/2} \, dx$

Thus,

$\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} \, dx$

$= -\sec\frac{x}{2} e^{-x/2} + \frac{1}{2}\int \sec\frac{x}{2}\tan\frac{x}{2} \, dx - \frac{1}{2}\int \sec\frac{x}{2}\tan\frac{x}{2} e^{-x/2} dx$

$= -\sec\frac{x}{2} e^{-x/2} + c$

In this unit we have exposed you to various methods of integration. You have also had a fair amount of practice in using these methods. We are now giving you some additional exercises. You may like to try your hand at these too. To solve these you will have to first identify the method which will suit the particular integrand the best. This is the crucial step. The next step where you apply the chosen method to get the answer is relatively easy. If you have studied this unit thoroughly, neither of these steps should pose any problem. So, good luck!

E15) Evaluate the following integrals :

a) $\int (2x^3+2x+3) \, dx$ b) $\int \frac{x^2+2}{x} \, dx$

c) $\int \sinh(x/2)\cosh(x/2) \, dx$

d) $\int (e^x - e^{-x})^2 \, dx$ e) $\int_2^4 \frac{x^2}{\sqrt{x^3+1}} \, dx$

f) $\int \frac{x}{(x^2+2)^n} \, dx$ g) $\int \sin x \, e^{\cos x} \, dx$

h) $\int \frac{1}{1+9x^2} \, dx$ i) $\int_0^{\pi/2} \frac{\sin x \cos x}{(1+\sin x)^3} \, dx$

j) $\int (x^2+2)^6 x^3\,dx$ k) $\int x\sqrt{x^4+2x^2+2}\,dx$

l) $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$ m) $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

n) $\int e^x(\ln \sin x + \cot x)\,dx$

E16) Prove that

$$\int u\frac{d^2v}{dx^2}\,dx = u\,\frac{dv}{dx} - v\,\frac{du}{dx} + \int v\,\frac{d^2u}{dx^2}\,dx,$$

and use it to evaluate $\int x^3 \sin x\, dx$

Before we end this unit, here are some general remarks about the existence of integrals. The result

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where F(x) is the antiderivative of f(x), will make sense only if f(x) exists at every point of the interval. Hence we have to be careful in using this result.

Thus,

$$\int \frac{1}{x}\,dx = [\ln|x|]_a^b = \ln \frac{|b|}{|a|}$$

But 1/x is not defined at x = 0, and ln |x| is also not differentiable at x = 0. As such, at this stage, we should use the result only if the interval [a,b] does not include x = 0.

Thus,

$$\int_{-1}^{2} \frac{1}{x}\,dx = \ln \frac{|2|}{|-1|} = \ln 2 \text{ is not valid.}$$

$$\int_{-2}^{-1} \frac{1}{x}\,dx = \ln \frac{|-1|}{|-2|} = \ln \frac{1}{2} \text{ is valid.}$$

Again, consider

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = [\sin^{-1}x]_0^1 = \frac{\pi}{2}.$$

We have used this in Example 3. However $\frac{1}{\sqrt{1-x^2}}$ does not exist at x = 1, and $\sin^{-1}x$ is not differentiable at x = 1. $L(\sin^{-1}x)'$ exists at x = 1, but $R(\sin^{-1}x)'$ does not exist, since $\sin^{-1}x$ itself does not exist when x > 1.

However, the above result is true in some sense. This sense will be clear to you in your course on analysis.

The antiderivative of every function need not exist, i.e., it need not be any of the functions we are familiar with. For example, there is no function known to us whose derivative is e^{-x^2}. However, the value of the definite integral $\int_a^b f(x)\,dx$ of every function, where f(x) is continuous on the interval [a,b], can be found out by numerical methods to any degree of approximation. You can study these methods in detail if you take the course on numerical analysis. You will study two simple numerical methods in Block 4 too. Thus, we cannot find the antiderivative of e^{-x^2}, but still, we can find the approximate value of

$\int_a^b e^{-x^2}\,dx$, for all real values of a and b. In fact, this integral is every important in probability theory and you will use it very often if you take the course on probability and statistics.

That brings us to the end of this unit. Let us summarise what we have studied so far.

11.5 SUMMARY

In this unit we have covered the following points.

1) If F(x) is an antiderivative of f(x), then the indefinite integral (or simply, integral) of f(x) is

$$\int f(x)\,dx = F(x) + c, \text{ where c is an arbitrary constant.}$$

2) $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots\ldots + k_n f_n(x)]\, dx =$

$k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + \ldots\ldots + k_n \int f_n(x)dx$

3) The method of substitution gives :

$$\int_a^b f[g(x)]\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du, \text{ if } u = g(x).$$

In particular,

$$\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1, \text{ and}$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln |f(x)| + c$$

$$\int_0^a f(x)\, dx = \int_0^{a/2} f(x)\, dx + \int_0^{a/2} f(a-x)\, dx$$

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f \text{ is even} \\ 0, & \text{if } f \text{ is odd.} \end{cases}$$

4) Standard formulas :

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln \frac{x + \sqrt{a^2+x^2}}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln \frac{x + \sqrt{x^2-a^2}}{a} + c$$

5) Integration of a product of two functions (integration by parts) :

$$\int u(x)v(x)dx = u(x) \int v(x)dx - \int \{u'(x) \int v(x)dx\}\, dx$$

This leads us to:

$$\int \sqrt{a^2-x^2}\, dx = \frac{1}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\int \sqrt{a^2+x^2}\, dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\ln \frac{x + \sqrt{a^2+x^2}}{a} + c$$

$$\int \sqrt{x^2-a^2}\, dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln \frac{x + \sqrt{x^2-a^2}}{a} + c$$

$$\int e^{ax}\sin bx\, dx = \frac{1}{\sqrt{a^2+b^2}}\, e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

$$\int e^{ax}\cos bx\, dx = \frac{1}{\sqrt{a^2+b^2}}\, e^{ax}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$$

11.6 SOLUTIONS AND ANSWERS

E 1) a) i) $\frac{x^5}{5} + c$ ii) $-2x^{-1/2} + c$ iii) $-4x^{-1} + c$

b) i) $x - x^2 + \frac{x^3}{3} + c$ ii) $\frac{x^3}{3} - 2x - \frac{1}{x} + c$

iii) $x + \frac{3x^2}{2} + x^3 + \frac{x^4}{4} + c$

c) i) $e^x - e^{-x} + 4x + c$ ii) $4\sin x + 3\cos x + e^x + \frac{x^2}{2} + c$

iii) $4\tanh x + e^x - 4x^2 + c$

d) i) $2\sin^{-1}x + 5\ln|x| + c$

ii) $\int \frac{2(x^2+1)+3}{x^2+1}\,dx = 2\int dx + 3\int \frac{1}{x^2+1}\,dx$

$= 2x + 3\tan^{-1}x + c$

e) i) $\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx + c$ ii) $\frac{x^2}{2} - 2x + \ln|x| + c$

f) i) $\int \frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x}\,dx$

$= \int \frac{(\sin^2 x+\cos^2 x)^2 - 2\sin^2 x\cos^2 x}{\sin^2 x \cos^2 x}\,dx$

$= \int \frac{1}{\sin^2 x}\,dx + \int \frac{1}{\cos^2 x}\,dx - 2\int dx$

$= -\cot x + \tan x - 2x + c$

iii) $6x + \frac{3x^2}{2} - \frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2} + c$

E 2) a) i) $\frac{6^5}{5} - 5^4$ ii) $\frac{1}{2} + \ln 2$

b) i) $\frac{275}{12}$ ii) $\frac{15}{4}$

E 3) a) $\int (5x-3)^{1/2}\,dx = \frac{1}{5}\int 5(5x-3)^{1/2}\,dx$ if $5x-3 = u, \frac{du}{dx} = 5$

$= \frac{1}{5}\int u^{1/2}\,du = \frac{1}{5}\,\frac{u^{3/2}}{3/2} + c = \frac{2}{15}(5x-3)^{3/2} + c$

b) $\frac{1}{14}(2x+1)^7 + c$ c) $\frac{1}{5}\ln\frac{19}{9}$ d) $\frac{1}{2}\ln|10x+7| + c$

e) $\frac{1}{2}\ln|x^2+2x+7| + c$ f) $\ln|x^3+x^2+x-8|\Big]_2^3 = \ln\frac{31}{6}$

g) $\frac{(3/4)(x^{4/3}-1)^{3/2}}{3/2} + c = \frac{1}{2}(x^{4/3}-1)^{3/2} + c$ h) $-\frac{1}{3}\sqrt{1-3x^2} + c$

E 4)

S.No.	f(x)	$\int f(x)\,dx$
1.	$\sin ax$	$-\frac{1}{a}\cos ax + c$
2.	$\cos ax$	$\frac{1}{a}\sin ax + c$
3.	$\sec^2 ax$	$\frac{1}{a}\tan ax + c$
4.	$\operatorname{cosec}^2 ax$	$-\frac{1}{a}\cot ax + c$
5.	$\sec ax \tan ax$	$\frac{1}{a}\sec ax + c$
6	$\operatorname{cosec} ax \cot ax$	$-\frac{1}{a}\operatorname{cosec} ax + c$
7.	e^{ax}	$\frac{1}{a}e^{ax} + c$
8.	a^{mx}	$\frac{1}{m}\frac{a^{mx}}{\ln a} + c$

E 5) a) $\int \sec x\, dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\, dx = \ln|\sec x+\tan x| + c$

b) $\int_0^{\pi/2} \sin^2 x \cos x\, dx = \frac{\sin^3 x}{3}\Big]_0^{\pi/2} = \frac{1}{3}$

c) If $u = \tan x$, $\frac{du}{dx} = \sec^2 x$

$\Rightarrow \int e^{\tan x} \sec^2 x\, dx = \int e^u\, du = e^u + c = e^{\tan x} + c$

E 6) a) i) $\frac{\sin 6x}{6} + c$ ii) $\frac{-2}{\sin^2 x} + c$

iii) $\int_{\pi/6}^{\pi/3} \cot 2x \operatorname{cosec}^2 2x\, dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} \cot 2x\, (2\operatorname{cosec}^2 2x)\, dx$

$= \frac{1}{2} \times \frac{\cot^2 2x}{2}\Big]_{\pi/6}^{\pi/3} = 0$

iv) Put $\cos 2\theta = u$. Then $\frac{du}{d\theta} = -2\sin 2\theta$

$\int \sin 2\theta\, e^{\cos 2\theta}\, d\theta = -\frac{1}{2} e^u\, du = -\frac{1}{2} e^u + c,$

$= -\frac{1}{2} e^{\cos 2\theta} + c$

v) $\int_0^{\pi/2} \sin\theta(1+\cos^4\theta)\, d\theta = \int_0^{\pi/2} \sin\theta\, d\theta + \int_0^{\pi/2} \sin\theta \cos^4\theta\, d\theta$

$= -\cos\theta\Big]_0^{\pi/2} - \frac{\cos^5\theta}{5}\Big]_0^{\pi/2}$

$= 1 + \frac{1}{5} = \frac{6}{5}$

b) i) $-\frac{(1+\cos\theta)^5}{5} + c.$

ii) $\frac{1}{10} \frac{1}{(1-5\tan\theta)^2}\Big]_0^{\pi/3}$

$= \frac{1}{2} \frac{2\sqrt{3}-15}{(1-5\sqrt{3})^2}$

iii) $\frac{(1+\sec\theta)^4}{4}\Big]_0^{\pi/4} = \frac{(1+\sqrt{2})^4 - 2^4}{4} = \frac{1+12\sqrt{2}}{4}$

c) i) $\int \sin^4\theta\, d\theta = \int \sin^3\theta \sin\theta\, d\theta$

$= \int (\frac{3}{4}\sin^2\theta - \frac{1}{4}\sin\theta \sin 3\theta)\, d\theta$

$\frac{3}{8}\int \{1 - (1-2\sin^2\theta)\}\, d\theta - \frac{1}{8}\int (\cos 2\theta - \cos 4\theta)\, d\theta$

$= \frac{3}{8}\int d\theta - \frac{3}{8}\int \cos 2\theta\, d\theta - \frac{1}{8}\int \cos 2\theta\, d\theta + \frac{1}{8}\int \cos 4\theta\, d\theta$

$= \frac{3}{8}\theta - \frac{1}{2}\frac{\sin 2\theta}{2} + \frac{1}{8}\frac{\sin 4\theta}{4} + c$

$= \frac{1}{4}(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta) + c$

ii) $\int \sin 3\theta \cos\theta\, d\theta = \frac{1}{2}[\int \sin 4\theta\, d\theta + \int \sin 2\theta\, d\theta]$

$$= \frac{1}{4}\left[\frac{-\cos 4\theta}{2} - \cos 2\theta + c\right]$$

iii) $$\int_0^{\pi/2} \cos 5\theta \cos\theta \, d\theta = \frac{\sin 4\theta}{8} - \frac{\sin 6\theta}{12}\Big]_0^{\pi/2} = 0$$

iv) $$\int_0^{\pi/2} \cos\theta \cos 2\theta \cos 4\theta \, d\theta = \frac{19}{105}$$

ii) $$\int \frac{dx}{\sqrt{1+x+x^2}} = \int \frac{dx}{\sqrt{(3/4)+(x+1/2)^2}} = \sinh^{-1}\left(\frac{x+(1/2)}{\sqrt{3}/2}\right) + c$$

$$\text{or} \quad \ln\left|\frac{(x+1/2)+\sqrt{3/4+(x+1/2)^2}}{\sqrt{3}/2}\right| + c$$

$$= \ln\left|\frac{(x+1/2)+\sqrt{x^2+x+1}}{\sqrt{3}/2}\right| + c$$

$$= \ln\left|\frac{2x+1+2\sqrt{x^2+x+1}}{\sqrt{3}}\right| + c$$

iii) $$\int \frac{dy}{\sqrt{y^2+6y+5}} = \int \frac{dy}{\sqrt{(y+3)^2-4}} = \cosh^{-1}\left(\frac{y+3}{2}\right) + c$$

iv) $$\int \frac{x^2}{1+x^2}\,dx = \int dx - \int \frac{1}{1+x^2}\,dx$$

$$= x - \tan^{-1}x + c$$

E7) $v = \int \frac{-500}{1+t^2}\,dt + c.$
$= -500\tan^{-1} t + c.$
$v(0) = 700 = -500\tan^{-1}0 + c = c$
$\Rightarrow c = 700.$
$v(3) = 700 - 500\tan^{-1}3.$

E8) For solution, see P. 104

E9) a) $$\int_0^{\pi} \sin^5 x \cos^3 x \, dx = \int_0^{\pi/2} \sin^5 x \cos^3 x \, dx + \int_0^{\pi/2} \sin^5(\pi-x)\cos^3(\pi-x)\, dx$$

$$= \int_0^{\pi/2} \sin^5 x \cos^3 x \, dx - \int_0^{\pi/2} \sin^5 x \cos^3 x \, dx = 0$$

b) $$\int_0^{\pi/2} \sin 2x \ln \tan x \, dx = \int_0^{\pi/4} \sin 2x \ln \tan x \, dx$$

$$+ \int_0^{\pi/4} \sin 2\left(\frac{\pi}{2} - x\right) \ln \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/4} \sin 2x \ln \tan x \, dx + \int_0^{\pi/4} \sin 2x \ln \cot x \, dx$$

$$= \int_0^{\pi/4} \sin 2x \ln(\tan x \cdot \cot x)\, dx$$

$$= \int_0^{\pi/4} \sin 2x \ln 1 \, dx = 0$$

c) $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^{a} f(x)\,dx$$

Put $x = -y$ in $\int_{-a}^{0} f(x)\,dx$

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

E10) a) $\int x^2 \ln x\,dx = \ln x \int x^2\,dx - \int \left(\frac{1}{x}\int x^2\,dx\right) dx$

$$= \ln x \cdot \frac{x^3}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$$

b) $xe^x + c$

c) $\int (1+x^2)\,e^x\,dx = (1+x^2)\,e^x - 2\int xe^x\,dx = (1+x^2)\,e^x - 2[xe^x - \int e^x\,dx]$

$$= (1+x^2)\,e^x - 2xe^x + 2e^x + c$$

$$= e^x\,(x^2-2x+3) + c$$

d) $\frac{1}{4}\left[-x^2\cos 2x + x\sin 2x + \frac{1}{2}\cos 2x\right] + c$

E11) a) $\int \sin^{-1}x\,dx = \sin^{-1}x \cdot x - \int \frac{1}{\sqrt{1-x^2}}\,x\,dx$

$$= x\sin^{-1}x + \sqrt{1-x^2} + c$$

b) $\frac{\pi}{4} - \frac{1}{2}\ln 2$

c) $x\cot^{-1}x + \frac{1}{2}\ln(1+x^2)$

E12) a) $\int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

Put $x = \sin u$ in $\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2 u}{\cos u}\cos u\,du$

$$= \int \sin^2 u\,du = \int \frac{1-\cos 2u}{2}\,du$$

$$= \frac{1}{2}u - \frac{1}{4}\sin 2u + c = \frac{1}{2}u - \frac{1}{2}\sin u\cos u + c$$

$$= \frac{1}{2}[\sin^{-1}x - x\cos(\sin^{-1}x)] + c$$

$\therefore \int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}[\sin^{-1}x - x\sqrt{1-x^2}] + c$

b) $\int \ln(1+x^2)\,dx = \int 1\cdot\ln(1+x^2)\,dx$

$$= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

$$= x\ln(1+x^2) - \int 2\left[1 - \frac{1}{1+x^2}\right] dx$$

$$= x\ln(1+x^2) - 2[x - \tan^{-1}x] + c$$

E13 a) $\frac{1}{20}e^{2x}(2\cos 4x + 4\sin 4x) + c$

b) $\frac{1}{18}e^{3x}(3\sin 3x - 3\cos 3x) + c$

c) $\int e^{4x}\cos x\cos 2x\,dx = \frac{1}{2}\int e^{4x}(\cos 3x + \cos x)\,dx$

$$= \frac{1}{2}\left[\int e^{4x}\cos 3x\,dx + \int e^{4x}\cos x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{25} e^{4x}(4\cos 3x + 3\sin 3x) + \frac{1}{17} e^{4x}(4\cos x + \sin x)\right] + c$$

d) $\int e^{2x}\cos^2 x\,dx = \int e^{2x}\left(\frac{\cos 2x+1}{2}\right) dx$

$$= \frac{1}{2}\left[\int e^{2x}\cos 2x\,dx + \int e^{2x}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{8} e^{2x}(2\cos 2x + 2\sin 2x) + \frac{1}{2} e^{2x}\right] + c$$

e) $\int \cosh ax \sin bx\,dx = \int \left(\frac{e^{ax}+e^{-ax}}{2}\right) \sin bx\,dx$

$$= \frac{1}{2}\left[\int e^{ax}\sin bx\,dx + \int e^{-ax}\sin bx\,dx\right]$$

$$= \frac{1}{2}\frac{1}{(a^2+b^2)}\left[e^{ax}(a\sin bx - b\cos bx) + e^{-ax}(-a\sin bx - b\cos bx)\right] + c$$

f) $\int xe^{ax}\sin bx\,dx = x\int e^{ax}\sin bx\,dx$

$$-\int \frac{1}{a^2+b^2} e^{ax}(a\sin bx - b\cos bx)\,dx$$

$$= \frac{x}{a^2+b^2} e^{ax}(a\sin bx - b\cos bx)$$

$$-\frac{1}{(a^2+b^2)^2}\left[ae^{ax}(a\sin bx - b\cos bx) - be^{ax}(a\cos bx + b\sin bx)\right] + c$$

E14) a) $\int \sqrt{a^2+x^2}\,dx = x.\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$

$$= x\sqrt{a^2+x^2} - \int \frac{a^2+x^2}{\sqrt{a^2+x^2}}\,dx + \int \frac{a^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} + a^2 \ln \frac{x + \sqrt{a^2+x^2}}{a} - \int \sqrt{a^2+x^2}\,dx + c$$

$$\therefore \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln \frac{x + \sqrt{a^2+x^2}}{a} + c$$

b) $\int \sqrt{x^2-a^2}\,dx = x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - \int \frac{a^2}{\sqrt{x^2-a^2}}\,dx$$

$$\therefore \int \sqrt{x^2-a^2}\,dx = \frac{1}{2} x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln \frac{x + \sqrt{x^2-a^2}}{a} + c$$

E15) a) $\frac{x^4}{2} + x^2 + 3x + c$ b) $\frac{x^2}{2} + 2\ln|x| + c$

c) $\frac{1}{2}\cosh x + c$ d) $\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2} + c$

e) $\frac{2}{3}\sqrt{x^3+1}\,\Big]_2^4 = \frac{2}{3}(\sqrt{65} - \sqrt{9})$

f) $\frac{-1}{2}\frac{(x^2+2)^{-7}}{7} + c$ g) $-e^{\cos x} + c$

h) $\frac{1}{3}\tan^{-1}(3x) + c$

i) $\int_0^{\pi/2} \frac{\sin x \cos x}{(1+\sin x)^3}\,dx = \int_0^1 \frac{t}{(1+t)^3}\,dt$

$= \int_0^1 \frac{dt}{(1+t)^2} - \int_0^1 \frac{dt}{(1+t)^3}$

$= \frac{-1}{1+t} + \frac{1}{2(1+t)^2}\Big]_0^1 = \frac{1}{8}$

j) $\int (x^2+2)^6 x^3\,dx = \frac{1}{2}\int t^6(t-2)\,dt$

$= \frac{1}{2}\left[\int t^7\,dt - 2\int t^6\,dt\right]$

$= \frac{t^8}{16} - \frac{t^7}{7} + c$

$= \frac{(x^2+2)^8}{16} - \frac{(x^2+2)^7}{7} + c$

k) $\int x\sqrt{x^4+2x^2+2}\,dx = \int x\sqrt{(x^2+1)^2+1}\,dx = \frac{1}{2}\int \sqrt{t^2+1}\,dt$

$= \frac{1}{4}t\sqrt{1+t^2} + \frac{1}{4}\sinh^{-1}t + c$

$= \frac{1}{4}(x^2+1)\sqrt{x^4+2x^2+2} + \frac{1}{4}\sinh^{-1}(x^2+1) + c$

l) $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx \quad = \int \theta\sin\theta\,d\theta$, if $x = \tan\theta$

$= -\theta\cos\theta + \int \cos\theta\,d\theta$ (integration by parts)

$= -\theta\cos\theta + \sin\theta + c$

m) Put $x = \tan\theta$ in $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

Answer $= 2[\theta\tan\theta + \ln|\cos\theta|] + c$

n) $\int e^x(\ln\sin x + \cot x)\,dx = \int e^x \ln\sin x\,dx + \int e^x\cot x\,dx$

$= \ln\sin x\, e^x - \int \cot x\, e^x\,dx + \int e^x\cot x\,dx$

$= e^x\ln\sin x.$

E16) $\int u\frac{d^2v}{dx^2}\,dx = u\frac{dv}{dx} - \int \frac{du}{dx}\cdot\frac{dv}{dx}\,dx$

$= u\frac{dv}{dx} - v\frac{du}{dx} + \int v\frac{d^2v}{dx^2}\,dx$

$\int x^3\sin x\,dx = \int x^3\frac{d^2}{dx^2}(-\sin x)\,dx$

$= -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx$

$= -x^3\cos x + 3x^2\sin x - 6\left[-x\cos x + \int\cos x\,dx\right]$

$= -x^3\cos x + 3x^2\sin x + 6(x\cos x - \sin x) + c$

UNIT 12 REDUCTION FORMULAS

Structure

12.1 INTRODUCTION

In the first two units of this block we have introduced the concept of a definite integral and have obtained the values of integrals of some standard forms. We have also studied two important methods of evaluating integrals, namely, the method of substitution and the method of integration by parts. In the solution of many physical or engineering problems, we have to integrate some integrands involving powers or products of trigonometric functions. In this unit we shall devise a quicker method for evaluating these integrals. We shall consider some standard forms of integrands one by one, and derive formulas to integrate them.

The integrands which we will discuss here have one thing in common. They depend upon an integer parameter. By using the method of integration by parts we shall try to express such an integral in terms of another similar integral with a lower value of the parameter. You will see that by the repeated use of this technique, we shall be able to evaluate the given integral.

Objectives

After reading this unit you should be able to derive and apply the reduction formulas for

- $\int x^n e^x dx$
- $\int \sin^n x dx$, $\int \cos^n x dx$, $\int \tan^n x dx$, etc.
- $\int \sin^m x \cos^n x dx$
- $\int e^{ax} \sin^n x dx$
- $\int \sinh^n x dx$, $\int \cosh^n x dx$

12.2 REDUCTION FORMULA

Sometimes the integrand is not only a function of the independent variable, but also depends upon a number n (usually an integer). For example, in $\int \sin^n x\, dx$, the integrand $\sin^n x$ depends on x and n. Similarly, in $\int e^x \cos mx\, dx$, the integrand $e^x \cos mx$ depends on x and m. The numbers n and m in these two examples are called parameters. We shall discuss only integer parameters here.

On integrating by parts we sometimes obtain the value of the given integral in terms of another similar integral in which the parameter has a smaller value. Thus, after a number of steps we might arrive at an integrand which can be readily evaluated. Such a process is called the **method of successive reduction**, and a formula connecting an integral with parameter n to a similar integral with a lower value of the parameter, is called a **reduction formula.**

Through a reduction formula we reduce the value of the parameter.

Definition 1: A formula of the form

$\int f(x,n)dx = g(x) + \int f(x,k)dx,$

where $k < n$, is called a reduction formula.

Consider the following example as an illustration.

Example 1 The integrand in $\int x^n e^x dx$ depends on x and also on the parameter n which is the exponent of x. Let

$I_n = \int x^n e^x dx.$

Integrating this by parts, with x^n as the first function and e^x as the second function gives us

$I_n = x^n \int e^x dx - \int (n x^{n-1} \int e^x dx) dx$

$\quad = x^n e^x - n \int x^{n-1} e^x dx$

Note that the integrand in the integral on the right hand side is similar to the one we started with. The only difference is that the exponent of x is $n-1$. Or, we can say that the exponent of x is reduced by 1. Thus, we can write

$I_n = x^n e^x - n I_{n-1}$ .... (1)

The formula (1) is a reduction formula. Now suppose we want to evaluate I_4, that is,

$\int x^4 e^x dx$. Using (1) we can write $I_4 = x^4 e^x - 4I_3$

$= x^4e^x - 4[x^3e^x - 3I_2]$ using (1) for I_3.

$= x^4e^x - 4x^3e^x + 12 I_2$

$= x^4e^x - 4x^3e^x + 12x^2e^x - 24 I_1$, using (1) for I_2

$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24 I_0$.

Now $I_0 = \int x^0 e^x dx = \int e^x dx = e^x + c$.

Thus, the method of successive reduction gives us

$\int x^4 e^x dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24 xe^x + 24e^x + c$

in five simple steps. You must have noted that we were saved from having to integrate by parts four times. This became possible because of formula (1). In this unit we shall derive many such reduction formulas.

These fall into three main categories according as the integrand

i) is a power of trigonometric functions,

ii) is a product of trigonometric functions, and

iii) involves hyperbolic functions.

We will take these up in the next three sections.

12.3 INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

There are many occasions when we have to integrate powers of trigonometric functions. In this section we shall indicate how to proceed in such cases.

12.3.1 Reduction Formulas for $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$

In this sub-section we will consider integrands which are powers of either sinx or cosx.

Let's take a power of sinx first. For evaluating $\int \sin^n x\,dx$, we write

$I_n = \int \sin^n x dx = \int \sin^{n-1} x \sin x\, dx$, if $n > 1$.

Taking $\sin^{n-1}x$ as the first function and sin x as the second and integrating by parts, we get

$$I_n = -\sin^{n-1}x \cos x - (n-1)\int \sin^{n-2}x \cos x\,(-\cos x)\,dx$$

$$= -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \cos^2 x\,dx$$

$$= -\sin^{n-1}x \cos x + (n-1)\left[\int \sin^{n-2}x\,(1-\sin^2 x)\,dx\right]$$

$$= -\sin^{n-1}x \cos x + (n-1)\left[\int \sin^{n-2}x\,dx - \int \sin^n x\,dx\right]$$

$$= -\sin^{n-1}x \cos x + (n-1)\left[I_{n-2} - I_n\right]$$

Hence,

$I_n + (n-1)I_n = -\sin^{n-1}x \cos x + (n-1)I_{n-2}$

That is, $nI_n = -\sin^{n-1}x \cos x + (n-1)I_{n-2}$. Or,

$$I_n = \frac{-\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

For n = 1
$\int \sin^n x\,dx = \int \sin x\,dx$
$= -\cos x + c$

This is the reduction formula for $\int \sin^n x dx$ (valid for $n \geq 2$).

Example 2 We will now use the reduction formula for $\int \sin^n x dx$ to evaluate the definite integral, $\int_0^{\pi/2} \sin^5 x dx$. We first observe that

$$\int_0^{\pi/2} \sin^n x dx = \left.\frac{-\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

$$= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x dx,\ n \geq 2.$$

Thus, $\int_0^{\pi/2} \sin^5 x dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x dx$

$$= \frac{4}{5}\cdot\frac{2}{3}\int_0^{\pi/2} \sin x dx$$

$$= \frac{8}{15}\left.(-\cos x)\right]_0^{\pi/2}$$

$$= \frac{8}{15}$$

Let us now derive the reduction formula for $\int \cos^n x dx$. Again, let us write

$I_n = \int \cos^n x dx = \int \cos^{n-1} x \cos x dx$, $n > 1$.

Integrating this integral by parts we get

$$I_n = \cos^{n-1}x \sin x - \int (n-1)\cos^{n-2}x\,(-\sin x).\,\sin x dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x \sin^2 x dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,(1-\cos^2 x)dx$$

$$= \cos^{n-1}x \sin x + (n-1)\,(I_{n-2} - I_n)$$

By rearranging the terms we get

$$I_n = \int \cos^n x dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

This formula is valid for $n \geq 2$. What happens when $n = 0$ or 1? You will agree that the integral in each case is easy to evaluate.

As we have observed in Example 2,

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx, \ n \geq 2.$$

Using this formula repeatedly we get

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x dx, \text{ if n is an odd number, } n \geq 3, \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} dx, \text{ if n is an even number, } n \geq 2. \end{cases}$$

This means

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3}, \text{ if n is odd, and } n \geq 3 \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2}, \text{ if n is even, } n \geq 2 \end{cases}$$

We can reverse the order of the factors, and write this as

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{2}{3} \frac{4}{5} \cdots \frac{n-3}{n-2} \frac{n-1}{n}, \text{ if n is odd, } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots \frac{n-1}{n} \frac{\pi}{2}, \text{ if n is even, } n \geq 2 \end{cases}$$

Arguing similarly for $\int_0^{\pi/2} \cos^n x dx$ we get

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{n-1}{n}, \text{ if n is odd, and } n \geq 3 \\ \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{n-1}{n} \cdot \frac{\pi}{2}, \text{ if n is even, } n \geq 2. \end{cases}$$

We are leaving the proof of this formula to you as an exercise. See E 1).

E E1) Prove that $\int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{2}{3} \frac{4}{5} \cdots \frac{n-1}{n}, \text{ if n is odd, } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots \frac{n-1}{n} \frac{\pi}{2}, \text{ if n is even, } n \geq 2. \end{cases}$

E2) Evaluate a) $\int_0^{\pi/2} \cos^5 x\,dx$, b) $\int_0^{\pi/2} \cos^6 x\,dx$, using the reduction formula derived in E 1).

12.3.2 Reduction Formulas for $\int \tan^n x\,dx$ and $\int \sec^n x\,dx$

In this sub-section we will take up two other trigonometric functions : tanx and secx. That is, we will derive the reduction formulas for $\int \tan^n x\,dx$ and $\int \sec^n x\,dx$. To derive a reduction formula for $\int \tan^n x\,dx$, $n > 2$, we start in a slightly different manner. Instead of writing $\tan^n x = \tan x \tan^{n-1} x$, as we did in the case of $\sin^n x$ and $\cos^n x$, we shall write $\tan^n x = \tan^{n-2} x \tan^2 x$. You will shortly see the reason behind this. So, we write

$$I_n = \int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx \qquad \ldots\ldots \quad (2)$$

You must have observed that the second integral on the right hand side is I_{n-2}. Now in the first integral on the right hand side, the integrand is of the form $[f(x)]^m \cdot f'(x)$

As we have seen in Unit 11,

$$\int [f(x)]^m f'(x)\,dx = \frac{[f(x)]^{m+1}}{m+1} + c$$

Thus, $\int \tan^{n-2} x \sec^2 x\,dx = \dfrac{\tan^{n-1} x}{n-1} + c$

Therefore, (2) gives $I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$.

Thus the reduction formula for $\int \tan^n x\, dx$ is

$$\int \tan^n x dx = I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}.$$

You must have now realised why we wrote $\tan^n x = \tan^2 x \tan^{n-2}x$

To derive the reduction formula for $\int \sec^n x dx$ $(n > 2)$, we first write $\sec^n x = \sec^{n-2}x \sec^2 x$, and then integrate by parts. Thus,

$$I_n = \int \sec^n x dx = \int \sec^{n-2}x \sec^2 x dx$$

$$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-3}x \cdot \sec x \tan^2 x\, dx$$

$$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x \tan^2 x\, dx$$

$$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x (\sec^2 x - 1)\, dx$$

$$= \sec^{n-2}x \tan x - (n-2)(I_n - I_{n-2})$$

After rearranging the terms we get

$$\int \sec^n x dx = I_n = \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

These formulas for $\int \tan^n x\, dx$ and $\int \sec^n x\, dx$ are valid for $n > 2$. For $n = 0, 1$ and 2, the integrals $\int \tan^n x\, dx$ and $\int \sec^n x dx$ can be easily evaluated. You have come across them in Units 10 and 11.

Example 3 Let's calculate i) $\int_0^{\pi/4} \tan^5 x\, dx$ and ii) $\int_0^{\pi/4} \sec^6 x\, dx$

i) $$\int_0^{\pi/4} \tan^5 x\, dx = \frac{\tan^4 x}{4}\Big]_0^{\pi/4} - \int_0^{\pi/4} \tan^3 x\, dx$$

$$= \frac{1}{4} - \frac{\tan^2 x}{2}\Big]_0^{\pi/4} + \int_0^{\pi/4} \tan x\, dx$$

$$= \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \frac{\sin x}{\cos x} dx$$

$$= -\frac{1}{4} - \ln(\cos x)\Big]_0^{\pi/4}$$

$$= -\frac{1}{4} - \ln\frac{1}{\sqrt{2}} + \ln 1$$

$$= -\frac{1}{4} + \ln\sqrt{2}$$

ii) $$\int_0^{\pi/4} \sec^6 x\, dx = \frac{\sec^4 x \tan x}{5}\Big]_0^{\pi/4} + \frac{4}{5}\int_0^{\pi/4} \sec^4 x\, dx$$

$$= \frac{4}{5} + \frac{4}{5}\left\{\frac{\sec^2 x \tan x}{3}\Big]_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2 x dx\right\}$$

$$= \frac{4}{5} + \frac{8}{15} + \frac{8}{15}\int_0^{\pi/4} \sec^2 x dx$$

$$= \frac{4}{3} + \frac{8}{15} \tan x\Big]_0^{\pi/4} = \frac{28}{15}$$

On the basis of our discussion in this section you will be able to solve these exercises.

E3) Derive the following reduction formulas for $\int \cot^n x\,dx$ and $\int \text{cosec}^n x\,dx$:

a) $\int \cot^n x\,dx = I_n = \frac{-1}{n-1}\cot^{n-1}x - I_{n-2}$

b) $\int \text{cosec}^n x\,dx = I_n = \frac{-\text{cosec}^{n-2}x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

E4) Evaluate

a) $\int_{\pi/4}^{\pi/2} \text{cosec}^3 x\,dx$ b) $\int_0^{\pi/2} \sin^8 x\,dx$ c) $\int \sec^3\theta\,d\theta$

12.4 INTEGRALS INVOLVING PRODUCTS OF TRIGONOMETRIC FUNCTIONS

In the last section we have seen the reduction formulas for the case where integrands were powers of a single trigonometric function. Here we shall consider some integrands involving products of powers of trigonometric functions. The technique of finding a reduction formula basically involves integration by parts. Since there can be more than one way of writing the integrand as a product of two functions, you will see that we can have many reduction formulas for the same integral. We start with the first one of the two types of integrands which we shall study in this section.

12.4.1 Integrand of the Type $\sin^m x \cos^n x$

The function $\sin^m x \cos^n x$ depends on two parameters m and n. To find a reduction formula for $\int \sin^m x \cos^n x dx$, let us first write

$$I_{m,n} = \int \sin^m x \cos^n x dx$$

Since we have two parameters here, we shall take a reduction formula to mean a formula connecting $I_{m,n}$ and $I_{p,q}$, where either $p < m$, or $q < n$, or both $p < m$, $q < n$ hold. In others words, the value of at least one parameter should be reduced.

$$\text{If } n = 1,\ I_{m,1} = \int \sin^m x \cos x\, dx = \begin{cases} \dfrac{\sin^{m+1}x}{m+1} + c, \text{ when } m \neq -1 \\ \ln|\sin x| + c, \text{ when } m = -1 \end{cases}$$

Hence we assume that $n > 1$. Now,

$$I_{m,n} = \int \sin^m x \cos^n x\, dx = \int \cos^{n-1}x\, (\sin^m x \cos x)\, dx$$

Integrating by parts we get

$$I_{m,n} = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} - \int (n-1)\cos^{n-2}x(-\sin x)\, \frac{\sin^{m+1}x}{m+1}\, dx, \text{ if } m \neq -1$$

$$= \frac{\cos^{n-1}x\, \sin^{m+1}x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2}x(1-\cos^2 x)dx$$

$$= \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1}\,[I_{m,n-2} - I_{m,n}]$$

Therefore,

$$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{m+n}{m+1} I_{m,n} = \frac{\cos^{n-1}x\, \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

This gives us,

$$I_{m,n} = \frac{\cos^{n-1}x\, \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{m,n-2} \quad \ldots\ldots \quad (3)$$

But, surely this formula will not work if $m + n = 0$. So, what do we do if $m + n = 0$? Actually we have a simple way out. If $m + n = 0$, then since n is positive, we write $m = -n$.

Remember we have taken $n > 1$?

Hence $I_{-n,n} = \int \sin^{-n}x \cos^n x dx = \int \cot^n x dx$, which is easy to evaluate using the reduction formula derived in Sec. 3 (See E2)).

To obtain formula (3) we had started with the assumption that $n > 1$. Instead of this, if we assume that $m > 1$, we can write

$I_{m,n} = \int \sin^m x \cos^n x dx = \int \sin^{m-1}x(\cos^n x \sin x)\, dx$. Integrating this by parts we get

$$I_{m,n} = \frac{-\sin^{m-1}x \cos^{n+1}x}{n+1} - (m-1)\int \sin^{m-2}x \cos x\, \frac{(-\cos^{n+1}x)}{m+n}\, dx \text{ for } n \neq -1$$

$$= \frac{-\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x\cos^{n}x(1-\sin^2x)dx$$

$$= \frac{-\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\left(I_{m-2,n} - I_{m,n}\right)$$

From this we obtain

$$I_{m,n} = \frac{-\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\left(I_{m-2,n}\right) \quad \ldots\ldots \quad (4)$$

If m or n is a positive odd integer, we can proceed as follows :

Suppose $n = 2p + 1$, $p > 0$, then

$$I_{m,n} = \int \sin^m x\cos^{2p+1}x\,dx = \int \sin^m x\,(1-\sin^2x)^p\cos x\,dx$$

$$= \int t^m(1-t^2)^p\,dt \text{ if we put } t = \sin x.$$

Expanding $(1-t^2)^p$ by binomial theorem and integrating term by term, we get

$$I_{m,n} = \frac{t^{m+1}}{m+1} - C(p,1)\frac{t^{m+3}}{m+3} - C(p,2)\frac{t^{m+5}}{m+5} \cdots + \frac{(-1)^p t^{m+2p+1}}{m+2p+1} + c$$

$$= \frac{\sin^{m+1}x}{m+1} - C(p,1)\frac{\sin^{m+3}x}{m+3} + C(p,2)\frac{\sin^{m+5}x}{m+5} - \cdots\cdots +$$

$$\frac{(-1)^p\sin^{m+2p+1}x}{m+2p+1} + c.$$

If m and n are positive integers, by repeated applications of formula (3) or formula (4), we keep reducing n or m by 2 at each step. Thus, eventually, we come to an integral of the form $I_{m,0}$ or $I_{m,1}$ or $I_{1,n}$ or $I_{0,n}$. In the previous section we have seen how these can be evaluated. This means we should be able to evaluate $I_{m,n}$ in a finite number of steps. We shall now look at an example to see how these formulas are used.

Example 4 Let us evaluate $\int_0^{\pi/2}\sin^4x\cos^6x\,dx$. Here $m = 4$ and $n = 6$. Since m is the smaller of the two, we shall employ formula (4) which reduces m at each step.

$$\int_0^{\pi/2}\sin^4x\cos^6x\,dx = \left[\frac{-\sin^3x\cos^7x}{10}\right]_0^{\pi/2} + \frac{3}{10}\int_0^{\pi/2}\sin^2x\cos^6x\,dx$$

$$= \frac{3}{10}\int_0^{\pi/2}\sin^2x\cos^6x\,dx$$

$$= \frac{3}{10}\left\{\left[\frac{-\sin x\cos^7x}{8}\right]_0^{\pi/2} + \frac{1}{8}\int_0^{\pi/2}\cos^6x\,dx\right\},$$

using formula (4) again.

$$= \frac{3}{80}\int_0^{\pi/2}\cos^6x\,dx$$

$$= \frac{3}{80} \times \frac{15\pi}{96} \text{ (from E2) b))} = \frac{3\pi}{512}$$

Are you ready to solve some exercises now?

E E5) In deriving formula (4) we had assumed that $m > 1$. How would you evaluate $I_{m,n}$ if $m = 1$?

E6) Formulas (3) and (4) fail when $m+n = 0$. We have seen how to evaluate $I_{m,n}$ if $m+n = 0$ and n is a positive integer. How would you evaluate it if $m+n = 0$ and n is a negative integer?

E7) Evaluate

a) $\int_0^{\pi/2} \sin^3 x \cos^5 x dx$ b) $\int_0^{\pi/2} \sin^8 x \cos^2 x dx$

12.4.2 Integrand of the Type $e^{ax}\sin^n x$

In this sub-section we will consider the evaluation of those integrals, where the integrand is a product of a power of a trigonometric function and an exponential function. That is, we will consider integrands of the type $e^{ax}\sin^n x$. Let us denote $\int e^{ax}\sin^n x\, dx$ by L_n, and integrate it by parts, taking $\sin^n x$ as the first function and e^{ax} as the second function. This gives us

$$L_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\int e^{ax}\sin^{n-1}x\cos x\, dx.$$

We shall now evaluate the integral on the right hand side, again by parts, with $\sin^{n-1}x \cos x$ as the first function and e^{ax} as the second one. Thus,

$$L_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a} - \frac{1}{a}\int e^{ax}\{(n-1)\sin^{n-2}x\cos^2 x - \sin^n x\}dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a} - \frac{1}{a}\int e^{ax}\{(n-1)\sin^{n-2}x - n\sin^n x\}dx\right]$$

$(n-1)\sin^{n-2}x\cos^2 x$
$= (n-1)\sin^{n-2}x\,(1-\sin^2 x)$
$= (n-1)\sin^{n-2}x - (n-1)\sin^n x.$

This means

$$L_n = \frac{e^{ax}\sin^n x}{a} - \frac{ne^{ax}\sin^{n-1}x\cos x}{a^2} + \frac{n(n-1)}{a^2}L_{n-2} - \frac{n^2}{a^2}L_n$$

Rearranging the terms we get

$$L_n = \frac{ae^{ax}\sin^n x}{n^2+a^2} - \frac{ne^{ax}\sin^{n-1}x\cos x}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2}L_{n-2}.$$

Given any L_n, we use this reduction formula repeatedly, till we get L_1 or L_0 (depending on whether n is odd or even). Since L_1 and L_0 are easy to evaluate, we are sure you can evaluate them yourself. (See E 8)). This means that L_n can be evaluated for any positive integer n.

Remark 1 If we put a = 0 in L_n, it reduces to the integral $\int \sin^n x dx$. This suggests that the reduction formula for $\int \sin^n x dx$ which we have derived in Sec. 3 is a special case of the reduction formula for L_n.

If you have followed the arguments in this sub-section closely, you should be able to do the exercises below.

E E8) Prove that

a) $L_0 = \frac{e^{ax}}{a} + c.$ b) $L_1 = \int e^{ax}\sin x dx = \frac{e^{ax}}{1+a^2}(a\sin x - \cos x) + c.$

E9) Prove : If $C_n = \int e^{ax}\cos^n x\,dx$, then

$$C_n = \frac{ae^{ax}\cos^n x}{n^2+a^2} + \frac{ne^{ax}\cos^{n-1}x\sin x}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2}C_{n-2}$$

E10) Verify that the reduction formula for $\int \cos^n x\,dx$ is a special case of the formula in E9).

12.5 INTEGRALS INVOLVING HYPERBOLIC FUNCTIONS

In this section we shall discuss the evaluation of integrals of the type $\int \sinh^n x\,dx$, $\int \cosh^n x\,dx$, etc.

Actually, you will find that the evaluation of these integrals does not involve any new techniques. In fact, the procedure we follow here is very similar to the one we followed for integrating $\sin^n x$, $\cos^n x$ etc. Let us find the reduction formula for, say, $\int \tanh^n x\,dx$. We are sure you will be able to follow this easily and derive the reduction formulas for the other hyperbolic functions (see E11)).

If $I_n = \int \tanh^n x\,dx$, we can write

$$I_n = \int \tanh^{n-2}x\tanh^2 x\,dx$$

$$= \int \tanh^{n-2}x\,(1-\text{sech}^2 x)dx$$

$$= \int \tanh^{n-2}x\,dx - \int \tanh^{n-2}x\,\text{sech}^2 x\,dx$$

$\frac{d}{dx}\tanh x = \text{sech}^2 x$

$$= I_{n-2} - \frac{\tanh^{n-1}x}{n-1}$$

Don't you agree that the above method is similar to the one adopted for $\int \tan^n x\, dx$? The following exercises can be easily done now.

E E11) Prove the following reduction formula:

$$\int \sinh^n x\, dx = \frac{\sinh^{n-1}x \cosh x}{n} - \frac{n-1}{n}\int \sinh^{n-2}x\, dx$$

E E12) Derive a reduction formula for $\int \cosh^n x dx$

That brings us to the end of this unit. We shall now summarise what we have covered in it.

12.6 SUMMARY

A reduction formula is one which links an integral dependent on a parameter with a similar integral with a lower value of the parameter.

In this unit we have derived a number of reduction formulas.

1 $\int x^n e^x\, dx = x^n e^x - n\int x^{n-1} e^x\, dx$

2 $\int \sin^n x\, dx = \frac{-\sin^{n-1}x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\, dx,\ n \geq 2$

3 $\int \cos^n x\, dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\, dx,\ n \geq 2$

4 $\int \tan^n x\, dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\, dx,\ n > 2$

5 $\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx,\ n > 2$

6 $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \begin{cases} \frac{2}{3}\,\frac{4}{5}\cdots\cdots\frac{n-1}{n}, \text{ if n is odd, } n \geq 3. \\ \frac{1}{2}\,\frac{3}{4}\cdots\cdots\frac{n-1}{n}\,\frac{\pi}{2}, \text{ if n is even, } n \geq 2. \end{cases}$

7 $\int \sin^m x\cos^n x\,dx = \frac{\cos^{n-1}x\sin^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx,\ n \geq 1$

$= \frac{-\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx,\ m \geq 1$

8 $\int e^{ax}\sin^n x\,dx = \frac{ae^{ax}\sin^n x}{n^2+a^2} - \frac{ne^{ax}\sin^{n-1}x\cos x}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2}\int e^{ax}\sin^{n-2}x\,dx$

9 $\int \tanh^n x\,dx = \frac{-\tanh^{n-1}x}{n-1} + \int \tanh^{n-2}x\,dx$

We have noted that the prime technique of deriving reduction formulas involves integration by parts. We have also observed that many more reduction formulas involving other trigonometric and hyperbolic functions can be derived using the same technique.

12.7 SOLUTIONS AND ANSWERS

E1) We have $\int_0^{\pi/2} \cos^n x\,dx = \left.\frac{\cos^{n-1}x\sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2}x\,dx,\ n \geq 2$

$$= \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2}x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4}x\,dx$$

$$= \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2}\cos^0 x\,dx, \text{ if n is even} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\int_0^{\pi/2}\cos x\,dx, \text{ if n is odd} \end{cases}$$

$$= \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \text{ if n is even} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3} \text{ if n is odd} \end{cases}$$

E2) a) $\int_0^{\pi/2} \cos^5 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$

b) $\int_0^{\pi/2} \cos^6 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$

E3) a) $I_n = \int \cot^n x\,dx = \int \cot^{n-2}x\,(\mathrm{cosec}^2 x - 1)\,dx \quad n \geq 2$

$$= \int \cot^{n-2}x\,\mathrm{cosec}^2 x\,dx - I_{n-2}$$

Therefore, $I_n = \frac{-\cot^{n-1}x}{n-1} - I_{n-2}$

b) $I_n = \int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^{n-2}x \operatorname{cosec}^2 x\,dx \quad n > 2$

$= -\operatorname{cosec}^{n-2}x \cot x - \int (n-2)\operatorname{cosec}^{n-2}x\cot^2 x\,dx$

$= -\operatorname{cosec}^{n-2}x \cot x - (n-2)\int \operatorname{cosec}^{n-2}x\,(\operatorname{cosec}^2 x - 1)\,dx$

$= -\operatorname{cosec}^{n-2}x \cot x - (n-2)\,I_n + (n-2)\,I_{n-2}$

$I_n = \frac{-\operatorname{cosec}^{n-2}x\cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

E4) a) $\int_{\pi/4}^{\pi/2} \operatorname{cosec}^3 x\,dx = \frac{-\operatorname{cosec}x\cot x}{2}\Big]_{\pi/4}^{\pi/2} + \frac{1}{2}\int_{\pi/4}^{\pi/2} \operatorname{cosec}x\,dx$

$= -\frac{1}{\sqrt{2}} + \frac{1}{2}\ln\tan\frac{x}{2}\Big]_{\pi/4}^{\pi/2}$

$= -\frac{1}{\sqrt{2}} + \frac{1}{2}(\ln 1 - \ln\tan\frac{\pi}{8})$

$= -\frac{1}{\sqrt{2}} - \frac{1}{2}\ln\tan\frac{\pi}{8}$

b) $\int_0^{\pi/2} \sin^8 x\,dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}$

c) $\int \sec^3\theta\,d\theta = \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta$

$= \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\ln(\sec x + \tan x) + c$

E5) If $m = 1$, $I_{m,n} = I_{1,n} = \int \sin x\cos^n x\,dx$

$= \begin{cases} -\frac{\cos^{n+1}x}{n+1} + c \text{ if } n \neq -1 \\ -\ln|\cos x| + c \text{ if } n = -1 \end{cases}$

E6) $m+n = 0 \Rightarrow n = -m \Rightarrow$ m is a positive integer.

$I_{m,n} = \int \sin^m x\cos^{-m}x\,dx = \int \frac{\sin^m x}{\cos^m x}\,dx = \int \tan^m x\,dx$

Now use the formula for $\int \tan^m x\,dx$.

E7) a) $\int_0^{\pi/2} \sin^3 x\cos^5 x\,dx = \frac{-\sin^2 x\cos^6 x}{8}\Big]_0^{\pi/2} + \frac{2}{8}\int_0^{\pi/2} \sin x\cos^5 x\,dx$

$= \frac{2}{8}\int_0^{\pi/2} \sin x\cos^5 x\,dx = \frac{-2}{8}\cdot\frac{\cos^6 x}{6}\Big]_0^{\pi/2} = \frac{1}{24}$

b) $\int_0^{\pi/2} \sin^8 x\cos^2 x\,dx = \frac{\cos x\sin^9 x}{10}\Big]_0^{\pi/2} + \frac{1}{10}\int_0^{\pi/2} \sin^8 x\,dx$

$= \frac{1}{10}\int_0^{\pi/2} \sin^8 x\,dx$

$= \frac{1}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{7\pi}{512}$

E8) a) $L_0 = \int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

b) $L_1 = \int e^{ax}\sin x\,dx = \frac{e^{ax}\sin x}{a} - \frac{1}{a}\int e^{ax}\cos x\,dx$

$$= \frac{e^{ax}\sin x}{a} - \frac{e^{ax}\cos x}{a^2} - \frac{1}{a^2}\int e^{ax}\sin x\,dx$$

$$\therefore \int e^{ax}\sin x\,dx = \frac{ae^{ax}\sin x}{1+a^2} - \frac{e^{ax}\cos x}{1+a^2} + c$$

$$= \frac{e^{ax}}{1+a^2}(a\sin x - \cos x) + c$$

E9) $C_n = \frac{e^{ax}}{a}\cos^n x + \frac{n}{a}\int e^{ax}\cos^{n-1}x\sin x\,dx$

$$= \frac{e^{ax}\cos^n x}{a} + \frac{n}{a}\left[\frac{e^{ax}\cos^{n-1}x\sin x}{a} + \frac{1}{a}\int e^{ax}\{(n-1)\cos^{n-2}x\sin^2 x - \cos^n x\}\,dx\right]$$

$$= \frac{e^{ax}\cos^n x}{a} + \frac{n}{a^2}e^{ax}\cos^{n-1}x\sin x + \frac{n}{a^2}\int e^{ax}\{(n-1)\cos^{n-2}x - n\cos^n x\}\,dx$$

$$= \frac{e^{ax}\cos^n x}{a} + \frac{n}{a^2}e^{ax}\cos^{n-1}x\sin x + \frac{n(n-1)}{a^2}C_{n-2} - \frac{n^2}{a^2}C_n$$

$$\therefore C_n = \frac{ae^{ax}\cos^n x}{n^2+a^2} + \frac{ne^{ax}\cos^{n-1}x\sin x}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2}C_{n-2}$$

E 10) Put $a = 0$ in the formula for C_n.

$$\therefore C_n = \int\cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int\cos^{n-2}x\,dx$$

which is the reduction formula for $\int\cos^n x\,dx$

E 11) $\int\sinh^n x\,dx = \int\sinh^{n-1}x\sinh x\,dx$

$$= \sinh^{n-1}x\cosh x - (n-1)\int\sinh^{n-2}x\cosh^2 x\,dx$$

$$= \sinh^{n-1}x\cosh x - (n-1)\int\sinh^{n-2}x(1+\sinh^2 x)\,dx$$

$$= \sinh^{n-1}x\cosh x - (n-1)I_{n-2} - (n-1)I_n$$

$$I_n = \frac{\sinh^{n-1}x\cosh x}{n} - \frac{n-1}{n}I_{n-2}$$

E 12) $I_n = \int\cosh^n x\,dx = \int\cosh^{n-1}x\cosh x\,dx$

$$= \cosh^{n-1}x\sinh x - (n-1)\int\cosh^{n-2}x\sinh^2 x\,dx$$

$$= \cosh^{n-1}x\sinh x - (n-1)\int\cosh^{n-2}x(\cosh^2 x - 1)\,dx$$

$$= \cosh^{n-1}x\sinh x - (n-1)I_n + (n-1)I_{n-2}$$

$$I_n = \frac{\cosh^{n-1}x\sinh x}{n} + \frac{n-1}{n}I_{n-2}$$

UNIT 13 INTEGRATION OF RATIONAL AND IRRATIONAL FUNCTIONS

Structure

13.1 INTRODUCTION

In the previous unit you have come across various methods of integration. This unit, which is the last one in this block, will complete the discussion of methods of integration in this course. Here we shall deal with the integration of rational functions in detail. The method, which we shall describe in Sec. 2, depends upon partial fraction decomposition with which you might be already familiar.

Later on in the unit we shall consider some simple types of irrational functions. But a full discussion of the integration of irrational functions is beyond the scope of this course. We end the unit by giving you a check list of points to be considered before deciding upon the method of integration for any given function. While going through this unit you will need to recall several standard forms like

$\int \frac{dx}{\sqrt{x^2+a^2}}$, $\int \sqrt{x^2+a^2}\,dx$ etc. which we have already covered in Unit 11.

Objectives

After reading this unit you should be able to

- recognise proper and improper rational functions
- integrate rational functions of a variable by using the method of partial fractions
- integrate certain types of rational functions of sinx and cosx
- evaluate the integrals of some specified types of irrational functions
- decide upon the method of integration to be used for integrating any given function.

13.2 INTEGRATION OF RATIONAL FUNCTIONS

We know by now that it is easy to integrate any polynomial function, that is, a function f given by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$. In this section we shall see how a rational function is integrated. But let us first define a rational function.

Definition 1 A function R is called a rational function if it is given by $R(x) = Q(x)/P(x)$, where $Q(x)$ and $P(x)$ are polynomials. R is defined for all x for which $P(x) \neq 0$.

If the degree of $Q(x)$ is less than the degree of $P(x)$, we say that $R(x)$ is a proper rational function. Otherwise, it is called an improper rational function. Thus

$f(x) = \frac{x+1}{x^2+x+2}$ is a proper rational function, and

$g(x) = \frac{x^3+x+5}{x-2}$ is an improper one.

$$\begin{array}{r|l} & x^2+3x+6 \\ x-2 & x^3+x+5 \\ & x^3-2x^2 \\ \hline & 3x^2+5 \\ & 3x^2-6x \\ \hline & 6x+5 \\ & 6x-12 \\ \hline & 17 \end{array}$$

But g(x) can also be written as

$g(x) = (x^2+3x+6) + \frac{17}{x-2}$ (by long division)

Here we have expressed g(x), which is an improper rational function, as the sum of a polynomial and a proper rational function. This can be done for any improper rational function. Thus, we can always write

an improper rational function	=	a polynomial	+	a proper rational function

As we have already observed, a polynomial can be easily integrated. This means that the problem of integrating an improper rational function is reduced to that of integrating a proper rational function. Therefore, it is enough to study the techniques of integrating proper rational functions. But first let's see whether you can identify proper rational functions.

E E1) Which of the following functions are proper rational functions? Write the improper ones as a sum of a polynomial and a proper rational function.

a) $\frac{x^3+1}{x^4+x+1}$ b) $\frac{x^2+x-3}{x^2+1}$ c) $\frac{x+8}{x^2+5x+18}$

d) $\frac{x^4+x^3-5}{x-2}$

13.2.1 Some Simple Rational Functions

Now we shall consider some simple types of proper rational functions, like

$\frac{1}{x-a}$, $\frac{1}{(x-b)^k}$ and $\frac{x-m}{ax^2+bx+c}$. Later you will see that any proper rational function can be written as a sum of these simple types of functions.

We shall illustrate the method of integrating these functions through some examples.

Example 1 The simplest proper rational function is of the type $\frac{1}{(x-a)}$. From Unit 10, we already know that

$$\int \frac{1}{(x-a)}\, dx = \ln|x-a| + c.$$

Example 2 Consider the function $f(x) = \frac{1}{(x+2)^4}$

To integrate this function we shall use the method of substitution which we have studied in Unit 11. Thus, if we put

$u = x+2$, $\frac{du}{dx} = 1$ and we can write

$$\int \frac{1}{(x+2)^4}\,dx = \int \frac{1}{u^4}\,du = \int u^{-4}\,du$$

$$= \frac{u^{-3}}{-3} + c = \frac{-1}{3(x+2)^3} + c$$

The next example is a little more complicated.

Example 3 Consider the function $f(x) = \frac{2x+3}{x^2-4x+5}$

This has a quadratic polynomial in the denominator.

Now $\int \frac{2x+3}{x^2-4x+5}\,dx$ can be written as $\int \frac{2x-4}{x^2-4x+5}\,dx + \int \frac{7}{x^2-4x+5}\,dx$

Perhaps you are wondering why we have split the integral into two parts. The reason for this break up is that now the integrand in the first integral on the right is of the form $\frac{g'(x)}{g(x)}$; and we know that

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + c.$$

Thus, $\int \frac{2x-4}{x^2-4x+5}\,dx = \ln|x^2-4x+5| + c_1$

To evaluate the second integral on the right, we write

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x^2-4x+4)+1}\,dx = \int \frac{1}{(x-2)^2+1}\,dx$$

Now, if we put $x-2 = u$, $\frac{du}{dx} = 1$ and

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1}u + c_2$$

$$= \tan^{-1}(x-2) + c_2$$

This implies,

$$\int \frac{2x+3}{x^2-4x+5}\,dx = \ln|x^2-4x+5| + 7\tan^{-1}(x-2) + c.$$

In the beginning of this sub-section we said that any proper rational function can be written as the sum of some functions of the type we considered in the three examples above. In the next sub-section we shall see how this is done. But try to solve an exercise before reading the next section. It will give you some practice in evaluating integrals of the types mentioned in this sub-section.

E

E2) Evaluate

a) $\int \frac{dx}{2x-3}$ b) $\int \frac{dt}{(t+5)^2}$ c) $\int \frac{2x+1}{x^2+8x+1}\,dx$ d) $\int \frac{4x+1}{x^2+x+2}\,dx$

13.2.2 Partial Fraction Decomposition

In school you must have studied the factorisation of polynomials. For example, we know that

$x^2-5x+6 = (x-2)(x-3)$

Here $(x-2)$ and $(x-3)$ are two linear factors of x^2-5x+6.

You must have also come across polynomials like x^2+x+1, which cannot be factorised into real linear factors. Thus, it is not always possible to factorise a given polynomial into linear factors. But any polynomial can, in principle, be factored into linear and quadratic factors. We shall not prove this statement here. It is a consequence of the Fundamental theorem of algebra which has been stated in Unit 11 of the Linear Algebra course. The actual factorisation of a polynomial may not be very easy to carry out. But, whenever we can factorise the denominator of a proper rational function, we can integrate it by employing the method of partial fractions. The following examples will illustrate this method.

Example 4 Let us evaluate $\int \frac{5x-1}{x^2-1}\,dx$. Here the integrand $\frac{5x-1}{x^2-1}$ is a proper rational function.

Its denominator x^2-1 can be factored into linear factors as : $x^2-1 = (x-1)(x+1)$

This suggests that we can write the decomposition of $\frac{5x-1}{x^2-1}$ into partial fractions as :

$$\frac{5x-1}{x^2-1} = \frac{5x-1}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

If we multiply both sides by $(x-1)(x+1)$, we get

$5x-1 = A(x+1) + B(x-1)$. That is,

$5x-1 = (A+B)x + A-B$.

By equating the coefficients of x we get $A+B = 5$.

Equating the constant terms on both sides we get $A-B = -1$.

Solving these two equations in A and B we get $A = 2$ and $B = 3$.

Thus, $\frac{5x-1}{x^2-1} = \frac{2}{x-1} + \frac{3}{x+1}$

Integrating both sides of this equation, we obtain,

$$\int \frac{5x-1}{x^2-1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3}{x+1}\,dx$$
$$= 2\ln|x-1| + 3\ln|x+1| + c$$

As you have seen, the most important step in the evaluation of $\int \frac{5x-1}{x^2-1}\,dx$ was the decomposition of the integrand into partial fractions. The procedure for finding the values of the two unknowns A and B, involved two simple simultaneous equations in two unknowns. But the higher the degree of the denominator, the more will be the number of unknowns, and it might be very tedious to find them. What can we do in such cases? There is a very simple way out.

In the equation

$5x-1 = A(x+1) + B(x-1)$, if we put $x = -1$, we get $-6 = -2B$, or $B = 3$. Similarly, if we put $x = 1$, we get $4 = 2A$ or $A = 2$. Isn't this a much simpler way of finding A and B?

Note that 1 and − 1 are the zeros of the denominator $x^2 - 1$.

Let's go on to our next example now.

Example 5 Suppose we want to integrate $\frac{2x^2+x-4}{x^3-x^2-2x}$.

We first observe that the denominator factors as $x(x+1)(x-2)$.

This means we can write

$$\frac{2x^2+x-4}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

Multiplying by x^3-x^2-2x we get

$2x^2+x-4 = (x+1)(x-2)A + Bx(x-2) + Cx(x+1)$

0, − 1 and 2 are the zeros of x^3-x^2-2x.

Now, if we put $x = 0$ in this equation, we get

$-4 = -2A$ or $A = 2$.

Putting $x = -1$ gives us $-3 = +3B$, or $B = -1$.

Putting $x = 2$, we get $6 = 6C$, or $C = 1$.

$$\text{Thus, } \int \frac{2x^2+x-4}{x^3-x^2-2x}\,dx = 2\int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{1}{x-2}\,dx$$
$$= 2\ln|x| - \ln|x+1| + \ln|x-2| + c.$$

Our next example illustrates the use of this method when the denominator has repeated linear factors.

Example 6 Take a look at the denominator of the integrand in $\int \frac{x}{x^3-3x+2}\,dx$.

It factors into $(x-1)^2(x+2)$. The linear factor $(x-1)$ is repeated twice in the decomposition of x^3-3x+2.

In this case we write

$$\frac{x}{x^3-3x+2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

from this point we proceed as before to find A, B and C. We get

$x = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

We put $x = 1$ and $x = -2$ and get $C = 1/3$ and $A = -2/9$

Then to find B, let us put any other convenient value, say $x = 0$.

This give us $\quad 0 = A - 2B + 2C$

$$\text{or, } 0 = \frac{-2}{9} - 2B + \frac{2}{3}.$$

This implies $B = 2/9$. Thus,

$$\int \frac{x}{x^3-3x+2}\,dx = \frac{-2}{9}\int \frac{1}{x+2}\,dx + \frac{2}{9}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{(x-1)^2}\,dx$$
$$= \frac{-2}{9}\ln|x+2| + \frac{2}{9}\ln|x-1| - \frac{1}{3}\left(\frac{1}{x-1}\right) + c$$
$$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c.$$

In our next example, we shall consider the case when the denominator of the integrand contains an irreducible quadratic factor (i.e. a quadratic factor which cannot be further factored into linear factors).

Example 7 To evaluate

$$\int \frac{6x^3-11x^2+5x-4}{x^4-2x^3+x^2-2x}\,dx,$$

we factorise $x^4-2x^3+x^2-2x$ as $x(x-2)(x^2+1)$. Then we write

$$\frac{6x^3-11x^2+5x-4}{x^4-2x^3+x^2-2x} = \frac{A}{x}+\frac{B}{x-2}+\frac{Cx+D}{x^2+1}$$

Thus,

$$6x^3-11x^2+5x-4 = A(x-2)(x^2+1)+Bx(x^2+1)+(Cx+D)x(x-2)$$

Next, we substitute $x=0$ and $x=2$ to get $A=2$ and $B=1$.

Then we put $x=1$ and $x=-1$ (some convenient values) to get $C=3$ and $D=-1$.

Thus $\int \frac{6x^3-11x^2+5x-4}{x^4-2x^3+x^2-2x}\,dx = 2\int \frac{1}{x}\,dx + \int \frac{1}{x-2}\,dx + \int \frac{3x-1}{x^2+1}\,dx$

$$= 2\ln|x| + \ln|x-2| + \frac{3}{2}\int \frac{2x}{x^2+1}\,dx - \int \frac{dx}{x^2+1}$$

$$= 2\ln|x| + \ln|x-2| + \frac{3}{2}\ln|x^2+1| - \tan^{-1}x + c.$$

Thus, you see, once we decompose our integrand, which is a proper rational function, into partial fractions, then the given integral can be written as the sum of some integrals of the type discussed in Examples 1, 2 and 3.

All the functions which we integrated till now were proper rational functions. Now we shall take up an example of an improper rational function.

Example 8 Let us evaluate $\int \frac{x^3+2x}{x^2-x-2}\,dx$.

Since the integrand is an improper rational function, we shall first write it as the sum of a polynomial and a proper rational function.

Thus

$$\frac{x^3+2x}{x^2-x-2} = x+1+\frac{5x+2}{x^2-x-2}$$

Therefore, $\int \frac{x^3+2x}{x^2-x-2}\,dx = \int x\,dx + \int dx + \int \frac{5x+2}{x^2-x-2}\,dx$

$$= \frac{x^2}{2} + x + \int \frac{5x+2}{x^2-x-2}\,dx$$

Now let us decompose $\frac{5x+2}{x^2-x-2}$ into partial fraction as

$$\frac{5x+2}{x^2-x-2} = \frac{5x+2}{(x-2)(x+1)} = \frac{A}{x-2}+\frac{B}{x+1}$$

$5x+2 = A(x+1)+B(x-2)$

If $x=-1$, we get $-3=-3B$, that is, $B=1$.

If $x=2$, we get $12=3A$, that is $A=4$.

Therefore, $\int \frac{5x+2}{x^2-x-2}\,dx = 4\int \frac{dx}{x-2} + \int \frac{dx}{x+1}$

$$= 4\ln|x-2| + \ln|x+1| + c$$

Hence $\int \frac{x^3+2x}{x^2-x-2}\,dx = \frac{x^2}{2} + x + 4\ln|x-2| + \ln|x+1| + c.$

Try to do the following exercise now. You will find that each integrand falls in one of the various types we have seen in Examples 1 to 8.

E3) Evaluate

a) $\int \frac{2}{x^2+2x}\,dx$ b) $\int \frac{x\,dx}{x^2-2x-3}$ c) $\int \frac{3x-13}{x^2+3x-10}\,dx$

d) $\int \frac{6x^2+22x-23}{(2x-1)(x^2+x-6)}\,dx$ e) $\int \frac{3x^3}{x^2+x-2}\,dx$ f) $\int \frac{x^2+x-1}{(x-1)(x^2-x+1)}\,dx$

g) $\int \frac{x^3-4x}{(x^2+1)^2}\,dx$

13.2.3 Method of Substitution

The method of partial fraction decomposition which we studied in the last sub-section can be applied to all rational functions. We can say this because as we have mentioned earlier, the Fundamental theorem of algebra guarantees the factorisation of any polynomial into linear and quadratic factors. But the actual process of factorising a polynomial is sometimes not quite simple. In such cases it would be a good idea to critically examine the integrand to check if the method of substitution can be applied. We will now give two examples to show how we can sometimes integrate a given rational function with the help of a suitable substitution.

Example 9 Suppose we want to integrate $\frac{1}{x(x^5+1)}$ with respect to x.

For this we write $\int \frac{dx}{x(x^5+1)} = \int \frac{x^4\,dx}{x^5(x^5+1)}$

Now let us write $x^5 = t$. Then $\frac{dt}{dx} = 5x^4$.

$$\int \frac{x^4\,dx}{x^5(x^5+1)} = \frac{1}{5}\int \frac{dt}{t(t+1)}$$

$$= \frac{1}{5}\int \left[\frac{1}{t} - \frac{1}{t+1}\right] dt$$

$$= \frac{1}{5}\ln\left|\frac{t}{t+1}\right| + c$$

$$= \frac{1}{5}\ln\left|\frac{x^5}{x^5+1}\right| + c$$

Example 10 Let us integrate $\frac{x^2-1}{x^4+x^2+1}$ w.r.t. x

$$\int \frac{x^2-1}{x^4+x^2+1}\,dx = \int \frac{(1-1/x^2)}{x^2+1+1/x^2}\,dx \quad \text{(division by } x^2\text{)}$$

$$= \int \frac{(1-1/x^2)}{(x+1/x)^2-1}\,dx$$

$$= \int \frac{dt}{t^2-1} \quad \text{if we put } t = x + \frac{1}{x}$$

$$= \frac{1}{2}\int\left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$$

$$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + c.$$

In Examples 9 and 10 you must have noted that the denominators of the integrands were not easily factorisable. The method of substitution provided an easier alternative. See if you can solve this exercise now.

E

E4) Integrate the following functions w.r.t. x

a) $\frac{x^2-1}{1+x^4}$ b) $\frac{1+x^2}{1+x^2+x^4}$

The exercises in this section have given you a fair amount of practice in integrating rational functions. In the next section we take up the case of rational trigonometric functions.

13.3 INTEGRATION OF RATIONAL TRIGONOMETRIC FUNCTIONS

You know that a polynomial in two variables x and y is an expression of the form

$$P(x,y) = \sum_{n=0}^{k} \sum_{m=0}^{p} a_{m,n} x^m y^n, \quad a_{m,n} \in R.$$

Accordingly, a polynomial in sinx and cosx is an expression of the form

$$P(\sin x,\cos x) = \sum_{n=0}^{k} \sum_{m=0}^{p} a_{m,n} \sin^m x \cos^n x, \quad a_{m,n} \in R$$

The integration of P(sinx, cosx) can be carried out easily as we have already integrated $\sin^m x \cos^n x$ in Unit 12. An expression, which is the ratio of two polynomials, P(sinx,cosx) and Q(sinx, cosx) is called a rational function of sinx and cosx. In this section we shall discuss the integration of some simple rational functions in sinx and cosx. We shall first indicate a general method for integrating these functions.

Let f(sinx, cosx) be a rational function in sinx and cosx. The first step in the evaluation of the integral of f is to make the substitution $\tan \frac{x}{2} = t$.

Thus, $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1+t^2}{2}$

Since $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2t}{1+t^2}$,

and $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$

we get,

$$\int f(\sin x, \cos x)\,dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\,dt$$

$$= \int F(t)\,dt,$$

where $F(t) = f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}$

is a rational function of t. Now we can use the method of partial fraction decomposition to integrate F(t). In principle then, we can integrate any rational function in sinx and cosx. But in actual practice we find that the rational function F(t) is often complicated, and it is not feasible to apply the method of partial fractions. In this unit, however, we shall restrict ourselves to a few simple rational functions only.

Example 11 Let us integrate $\frac{1}{a+b\cos x}$.

Now $a+b\cos x = a(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}) + b(\cos^2\frac{x}{2} - \sin^2\frac{x}{2})$

$$= (a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}$$

Therefore, $\int\frac{dx}{a+b\cos x} = \int\frac{\sec^2\frac{x}{2}\,dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$

$$= \int\frac{\sec^2\frac{x}{2}\,dx}{(a-b)\left[\frac{a+b}{a-b}+\tan^2\frac{x}{2}\right]}$$

If we put $\tan\frac{x}{2} = t$, we get

$$\int\frac{dx}{a+b\cos x} = 2\int\frac{dt}{(a-b)\left(\frac{a+b}{a-b}+t^2\right)}$$

$$= \frac{2}{a-b}\int\frac{dt}{\frac{a+b}{a-b}+t^2}$$

If $a > b > 0$, then $\frac{a+b}{a-b} > 0$, and we get

$$\int\frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right)$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

If $0 < a < b$, then $\frac{a+b}{a-b} < 0$, and

$$\int\frac{dx}{a+b\cos x} = \frac{2}{\sqrt{b^2-a^2}}\ln\frac{\sqrt{b+a}+\sqrt{b-a}\,t}{\sqrt{b+a}-\sqrt{b-a}\,t}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\ln\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}$$

$1+\cos x = 2\cos^2\frac{x}{2}$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

Example 12 To evaluate $\int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx$, we write

$$\int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx = \int \frac{dx}{\sin x(1+\cos x)} + \int \frac{dx}{1+\cos x}$$

$$= \frac{1}{4}\int \frac{dx}{\sin\frac{x}{2}\cos^3\frac{x}{2}} + \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2}}$$

$$= \frac{1}{4}\int \frac{\sec^4\frac{x}{2}}{\tan\frac{x}{2}}\,dx + \frac{1}{2}\int \sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{2}\int \frac{1+t^2}{t}\,dt + \int dt \qquad (\tan\frac{x}{2} = t)$$

$$= \frac{1}{2}\left[\int \frac{1}{t}\,dt + \int t\,dt\right] + \int dt$$

$$= \frac{1}{2}\left[\ln|t| + \frac{t^2}{2}\right] + t + c.$$

Thus, $\int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx = \frac{1}{2}\ln|\tan x/2| + \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + c$

Now proceeding exactly as in Examples 11 and 12, you can do these exercises.

E

E5) Evaluate $\int \frac{dx}{a+b\sin x}$

E

E6) Integrate a) $\frac{1}{4+5\cos x}$ b) $\frac{\cos x}{2-\cos x}$ w.r.t. x.

By now you have seen and applied many different methods of integration. The crux of the matter lies in choosing the appropriate method for integrating a given function.

For example, suppose we ask you to integrate the function $\frac{\sin x \cos x}{1+\sin^2 x}$. Realising that this is a rational function in sinx and cosx, you may put $\tan\frac{x}{2} = t$ and proceed :

$$\int \frac{\sin x \cos x}{1+\sin^2 x}\,dx = 4\int \frac{t(1-t^2)\,dt}{(1+t^2)(1+6t^2+t^4)}$$

Now $1+6t^2+t^4 = (3+\sqrt{8}+t^2)(3-\sqrt{8}+t^2)$

By this step you will realise that it is going to be a tough job. But don't worry. There is an easy way out.

In $\int \frac{\sin x \cos x}{1+\sin^2 x}\,dx$, if we make the substitution $1+\sin^2 x = t$, we get

$$\int \frac{\sin x \cos x\,dx}{1+\sin^2 x} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + c.$$

$$= \frac{1}{2}\ln(1+\sin^2 x) + c.$$

Thus, the choice of the method is very crucial. And only practice can help you make a good choice.

We shall now illustrate some techniques used in integrating irrational functions.

13.4 INTEGRATION OF IRRATIONAL FUNCTIONS

The task of integrating functions gets tougher if the given function is an irrational one, that is, it is not of the form $\frac{Q(x)}{P(x)}$. In this section we shall give you some tips for evaluating some particular types of irrational functions. In most cases our endeavour will be to arrive at a rational function through an appropriate substitution. This rational function can then be easily evaluated by using the techniques developed in Sec. 2.

I) Integration of functions containing only fractional powers of x :

In this case we put $x = t^n$, where n is the lowest common multiple (l.c.m.) of the denominators of powers of x. This substitution reduces the function to a rational

function of t.
Look at the following example.

Example 13 Let us evaluate $\int \frac{2x^{1/2} + 3x^{1/3}}{1+x^{1/3}}\,dx$.

We put $x = t^6$, as 6 is the l.c.m. of 2 and 3. We get

$$\int \frac{2x^{1/2} + 3x^{1/3}}{1+x^{1/3}}\,dx = 6\int \frac{2t^3 + 3t^2}{1+t^2}\,t^5\,dt$$

$$= 6\int \frac{2t^8 + 3t^7}{1+t^2}\,dt = 6\int \left[2t^6+3t^5-2t^4-3t^3+2t^2+3t-2-\frac{3t-2}{1+t^2}\right]dt$$

$$= 6\left[\frac{2}{7}t^7 + \frac{1}{2}t^6 - \frac{2}{5}t^5 - \frac{3}{4}t^4 + \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t - \frac{3}{2}\ln(1+t^2) + 2\tan^{-1}t\right] + c$$

$$= \frac{12}{7}x^{7/6} + 3x - \frac{12}{5}x^{5/6} - \frac{9}{2}x^{2/3} + 4x^{1/2} + 9x^{1/3} - 12x^{1/6} - 9\ \ln\ |1+x^{1/3}| +$$

$$12\tan^{-1}x^{1/6} + c$$

II) Integral of the type $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Here we shall have to consider two cases : (i) $a > 0$ and (ii) $a < 0$.
In each case we will try to put the given integrand in a form which we have already seen how to integrate.

i) $a > 0$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{x^2+bx/a+c/a}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{(x+b/2a)^2 + c/a - b^2/4a^2}}$$

If we put $t = x+b/2a$, we get

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2 + (c/a - b^2/4a^2)}}$$

This is one of the standard types of integrals listed in Table 3 in Unit 11, and can thus be evaluated.

ii) $a < 0$: If we put $-a = d$, then $d > 0$, and we can write

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{d}}\int \frac{dx}{\sqrt{(c/d + b^2/4d^2) - (x - b/2d)^2}}$$

$$= \frac{1}{\sqrt{d}}\int \frac{dt}{\sqrt{(c/d + b^2/4d^2) - t^2}}, \quad \text{if } t = x - b/2d$$

This is again in one of the standard forms.

III) Integration of $\frac{1}{(fx+e)\sqrt{ax^2+bx+c}}$.

We will illustrate the method through an example.

Example 14 Suppose we want to evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+4x+2}}$.

Let us put $x+1 = 1/y$. Then $-\frac{1}{y^2}\frac{dy}{dx} = 1$.

Now we will try to express x^2+4x+2 in terms of y.
For this we write

$$x^2+4x+2 = (x+1)^2 + 2(x+1) - 1$$

$$= \frac{1}{y^2} + \frac{2}{y} - 1 = \frac{1+2y-y^2}{y^2}$$

Therefore,

$$\int \frac{dx}{(x+1)\sqrt{x^2+4x+2}} = \int \frac{\frac{-1}{y^2}\,dy}{\frac{1}{y}\sqrt{\frac{1+2y-y^2}{y^2}}} = -\int \frac{dy}{\sqrt{1+2y-y^2}}$$

$$= -\int \frac{dy}{\sqrt{2-(y-1)^2}} = \cos^{-1}\left(\frac{y-1}{\sqrt{2}}\right)$$

$$= \cos^{-1}\left[\frac{-x}{(x+1)\sqrt{2}}\right] + c.$$

This example suggests that in integrating $\frac{1}{(fx+e)\sqrt{ax^2+bx+c}}$, we should make the substitution $fx + e = \frac{1}{y}$, and then simplify the expression.

Let us move over to the next type now.

IV) Integration of $\frac{(Ax+B)}{\sqrt{ax^2+bx+c}}$

We break $Ax+B$ into two parts such that the first part is a constant multiple of the differential coefficient of ax^2+bx+c, that is, $2ax + b$, and the second part is independent of x. Thus,

$$Ax+B = \frac{A}{2a}(2ax+b) + B - \frac{Ab}{2a} \text{ and}$$

$$\int \frac{(Ax+B)\,dx}{\sqrt{ax^2+bx+c}} = \frac{A}{2a}\int \frac{(2ax+b)\,dx}{\sqrt{ax^2+bx+c}} + \frac{(2aB-Ab)}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= \frac{A}{a}\sqrt{ax^2+bx+c} + \frac{(2aB-Ab)}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Evaluation of the last integral has already been discussed in II).

V) Integration of $(Ax+B)\sqrt{ax^2+bx+c}$

We break $Ax+B$ as we did in IV), and obtain

$$\int (Ax+B)\sqrt{ax^2+bx+c}\,dx = \frac{A}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \frac{B2a-Ab}{2a}\int \sqrt{ax^2+bx+c}\,dx$$

$$= \frac{A}{3a}(ax^2+bx+c)^{3/2} + \frac{2aB-Ab}{2a}\int \sqrt{ax^2+bx+c}\,dx.$$

We have already seen how to evaluate the integral on the right hand side (see Sec. 4, Unit 11).

Let us use these methods to solve some examples now.

Example 15 To evaluate $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$,

we note that $x+2 = \frac{1}{2}(2x+2) + 1$, and write

$$\int \frac{(x+2)\,dx}{\sqrt{x^2+2x+3}} = \frac{1}{2}\int \frac{(2x+2)\,dx}{\sqrt{x^2+2x+3}} + \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$$= \sqrt{x^2+2x+3} + \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$$= \sqrt{x^2+2x+3} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c.$$

$$\frac{1}{\sqrt{x^2+2x+3}} = \frac{1}{\sqrt{(x+1)^2+2}}$$

Example 16 To evaluate $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$

we note that $x^2+2x+3 = x^2+x+1+x+2 = x^2+x+1+\frac{1}{2}(2x+1)+\frac{3}{2}$

Hence

$$\int \frac{(x^2+2x+3)}{\sqrt{x^2+x+1}}\,dx = \int \sqrt{x^2+x+1}\,dx + \frac{1}{2}\int \frac{(2x+1)}{\sqrt{x^2+x+1}}\,dx + \frac{3}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$$

$$= \int \sqrt{(x+\tfrac{1}{2})^2+\tfrac{3}{4}}\,dx + \sqrt{x^2+x+1}\,dx + \frac{3}{2}\ln\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+c$$

We have used two results from Unit 11 here.

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+c,$

and

$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+c$

$$= \frac{(x+\frac{1}{2})}{2}\sqrt{x^2+x+1} + \frac{3}{8}\ln\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right) + \sqrt{x^2+x+1} + \frac{3}{2}\ln\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+c$$

$$= \frac{1}{4}(2x+5)\sqrt{x^2+x+1} + \frac{15}{8}\ln\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+c.$$

See, if you can solve this exercise.

E E7) Integrate the following:

a) $\frac{\sqrt{x}}{1+\sqrt[4]{x}}$ b) $\frac{1}{(2-x)\sqrt{1-2x+3x^2}}$

When you are faced with a new integrand, the following suggestions furnish a thread through the labyrinth of methods.

(1) Check the integrand to see if it fits one of the patterns

$\int u^n du$ or $\int \frac{du}{u^n}$

(2) See if the integrand fits any one of the patterns obtained by the reversal of differentiation formulas. (We have considered these in Unit 11).

(3) If none of these patterns is appropriate, and if the integrand is a rational function, then our theory of partial fractions enables us to integrate it.

(4) If the integrand is a rational function of sinx and cosx, and simpler methods of previous units fail, the substitution $t = \tan\frac{x}{2}$ will make the integrand into a rational function of t, which can then be evaluated.

(5) If the integrand is a radical of one of the forms $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$, then the trigonometric substitutions $x = a\sin\theta$, $x = a\cos\theta$ or $x = a\sec\theta$ will reduce the integrand to a rational function of $\sin\theta$ and $\cos\theta$. If the radical is of the form $\sqrt{ax^2+bx+c}$, a square completion $\sqrt{a(x+b/2a)^2 + c-b^2/4a}$ will reduce it essentially to one of the above radicals.

(6) If the integrand is an irrational function of x, try to express it as a rational function or an integrable radical through appropriate substitutions.

(7) Inspect the integrand to see if it will yield to integration by parts.

Finally, we would like to remind you again that a lot of practice is essential if you want to master the various techniques of integration. We have already mentioned that a proper choice of the method of integration is the key to the correct evaluation of any integral. Now let us briefly recall what we have covered in this unit.

13.5 SUMMARY

In this unit we have covered the following points :

1 A rational function f of x is given by $f(x) = P(x)/Q(x)$, where P(x) and Q(x) are polynomials in x. It is called proper if the degree of P(x) is less than the degree of Q(x). Otherwise it is called improper.

2 A proper rational expression can be resolved into partial fractions with linear or quadratic denominators.

3 A rational function can be integrated by the method of partial fractions.

4 Integration of a rational function of sinx and cosx can be done by putting $t = \tan\frac{x}{2}$.

5 Integration of Irrational functions of the following types is discussed.

i) integrand contains fractional power of x.

ii) $\frac{1}{\sqrt{ax^2+bx+c}}$ iii) $\frac{1}{(fx+e)\sqrt{ax^2+bx+c}}$

iv) $\frac{Ax+B}{\sqrt{ax^2+bx+c}}$ v) $(Ax+B)\sqrt{ax^2+bx+c}$

6 A check list of points to be considered while evaluating any integral is given.

13.6 SOLUTIONS AND ANSWERS

E1) a) and c) are proper.

b) $\frac{x^2+x-3}{x^2+1} = 1 + \frac{x-4}{x^2+1}$

d) $\frac{x^4+x^3-5}{x-2} = x^3+3x^2+6x+12+\frac{19}{x-2}$

E2) a) $\int \frac{dx}{2x-3} = \frac{1}{2}\int \frac{2dx}{2x-3} = \frac{1}{2}\ln|2x-3| + c$

b) $\int \frac{dt}{(t+5)^2} = \frac{-1}{t+5} + c$

c) $\int \frac{2x+1}{x^2+8x+1}\,dx = \int \frac{2x+8}{x^2+8x+1}\,dx - 7\int \frac{dx}{x^2+8x+1}$

$= \ln|x^2+8x+1| - 7\int \frac{dx}{(x+4)^2-15}$

$= \ln|x^2+8x+1| - 7\int \frac{du}{u^2-15}$, if $u = x+4$.

$= \ln|x^2+8x+1| - \frac{7}{2\sqrt{15}}\ln\left|\frac{u-\sqrt{15}}{u+\sqrt{15}}\right| + c$

$= \ln|x^2+8x+1| - \frac{7}{2\sqrt{15}}\ln\left|\frac{x+4-\sqrt{15}}{x+4+\sqrt{15}}\right| + c$

d) $\int \frac{4x+1}{x^2+x+2}\,dx = \int \frac{2(2x+1)-1}{x^2+x+2}\,dx$

$= 2\int \frac{2x+1}{x^2+x+2}\,dx - \int \frac{dx}{x^2+x+2}$

$= 2\ln|x^2+x+2| - \int \frac{dx}{(x+\frac{1}{2})^2+\frac{7}{4}}$

$= 2\ln|x^2+x+2| - \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{7}/2}\right) + c$

$= 2\ln|x^2+x+2| - \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + c$

E3) a) $\frac{2}{x^2+2x} = \frac{2}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$\Rightarrow 2 = A(x+2) + Bx$

$x = 0 \Rightarrow 2 = 2A \Rightarrow A = 1$

$x = -2 \Rightarrow 2 = -2B \Rightarrow B = -1$

$\therefore \frac{2}{x^2+2x} = \frac{1}{x} - \frac{1}{x+2}$

$\therefore \int \frac{2}{x^2+2x}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+2}\,dx$

$= \ln|x| - \ln|x+2| + c = \ln\left|\frac{x}{x+2}\right| + c$

b) $\frac{x}{x^2-2x-3} = \frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$x = A(x+1) + B(x-3)$

$x = 3 \Rightarrow 3 = 4A \Rightarrow A = \frac{3}{4}$

$x = -1 \Rightarrow -1 = -4B \Rightarrow B = \frac{1}{4}$

$\therefore \int \frac{x}{x^2-2x-3}\,dx = \int \frac{3dx}{4(x-3)} + \int \frac{dx}{4(x+1)}$

$= \frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + c$

c) $\frac{3x-13}{x^2+3x-10} = \frac{3x-13}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$

$\therefore\ 3x - 13 = A(x-2) + B(x+5)$

$x = 2 \implies -7 = 7B \implies B = -1$

$x = -5 \implies -28 = -7A \implies A = 4$

$\implies \int \frac{3x-13}{x^2+3x-10}\,dx = 4\int \frac{dx}{x+5} - \int \frac{dx}{x-2}$

$= 4\ \ln|x+5| - \ln|x-2| + c$

d) $\frac{6x^2+22x-23}{(2x-1)(x^2+x-6)} = \frac{6x^2+22x-23}{(2x-1)(x+3)(x-2)} = \frac{A}{2x-1} + \frac{B}{x+3} + \frac{C}{x-2}$

$6x^2+22x-23 = A(x+3)(x-2) + B(x-2)(2x-1) + C(2x-1)(x+3)$

$x = 2 \implies 45 = 15C \implies C = 3$

$x = -3 \implies -35 = 35B \implies B = -1$

$x = 1/2 \implies \frac{-21}{2} = \frac{-21}{4}A \implies A = 2$

$\therefore \int \frac{6x^2+22x-23}{(2x-1)(x^2+x-6)}\,dx = \frac{1}{2}\ln|2x-1| - \ln|x+3| + 3\ \ln|x-2| + c$

e) $\frac{3x^3}{x^2+x-2} = 3x - 3 + \frac{9x-6}{x^2+x-2}$

$\therefore \int \frac{3x^3}{x^2+x-2}\,dx = \int (3x-3)\,dx + 3\int \frac{3x-2}{x^2+x-2}\,dx$

$= \frac{3x^2}{2} - 3x + 8\ln|x+2| + \ln|x-1| + c$

f) $\frac{x^2+x-1}{(x-1)(x^2-x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2-x+1}$

$\therefore\ x^2+x-1 = A(x^2-x+1) + (Bx+C)(x-1)$

$x = 1 \implies 1 = A$

$\therefore$ we have

$x^2 + x - 1 = x^2 - x + 1 + Bx^2 + (C-B)x - C$

Thus $1 = 1 + B$ (Coefficients of x^2)

$\therefore\ B = 0$

Also, $-1 = 1 - C$ (constant terms)

$\therefore\ C = 2$

$\therefore \int \frac{x^2+x-1}{(x-1)(x^2-x+1)}\,dx = \int \frac{dx}{x-1} + 2\int \frac{dx}{x^2-x+1}$

$= \ln|x-1| + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + c$

g) $\frac{x^3-4x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

$\therefore\ x^3 - 4x = (Ax+B)(x^2+1) + Cx + D$

$\therefore\ x^3 - 4x = Ax^3 + Bx^2 + (A+C)x + (B+D)$

$\therefore\ A = 1, B = 0, C = -5, D = 0$

$\therefore \int \frac{x^3-4x}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx - 5\int \frac{x}{(x^2+1)^2}\,dx$

$$= \frac{1}{2}\ln(x^2+1) + \frac{5}{2}\frac{1}{(x^2+1)} + c$$

E4) a) $\int \frac{x^2-1}{1+x^4}\,dx = \int \frac{1-\frac{1}{x^2}}{\frac{1}{x^2}+x^2}\,dx$

$$= \int \frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2-2}\,dx$$

$$= \int \frac{dt}{t^2-2} \qquad \text{if } t = x+\frac{1}{x}$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + c$$

b) $\int \frac{1+x^2}{1+x^2+x^4}\,dx = \int \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+1+x^2}\,dx$

$$= \int \frac{\frac{1}{x^2}+1}{(x-\frac{1}{x})^2+3}\,dx$$

$$= \int \frac{dt}{t^2+3}, \quad \text{if } t = x-\frac{1}{x},\ \frac{dt}{dx} = 1+\frac{1}{x^2}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{1}{\sqrt{3}}(x-\frac{1}{x})\right\} + c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + c$$

E5) $\int \frac{dx}{a+b\sin x} = \int \frac{2dt}{a(1+t^2)+2bt}, \quad \text{if } t = \tan x/2$

$$= 2\int \frac{2dt}{at^2+2bt+a} = 2\int \frac{dt}{(\sqrt{a}t+\frac{b}{\sqrt{a}})^2+(\frac{a^2-b^2}{a})}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{at+b}{\sqrt{a^2-b^2}}\right) + c$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{a\tan\frac{x}{2}+c}{\sqrt{a^2-b^2}}\right) + c$$

E6) a) $\int \frac{dx}{4+5\cos x} = 2\int \frac{dt}{\left\{4+5\left(\frac{1-t^2}{1+t^2}\right)\right\}(1+t^2)}$

$$= 2\int \frac{dt}{4+4t^2+5-5t^2} = 2\int \frac{dt}{9-t^2}$$

$$= \frac{1}{3}\ln\left|\frac{3+t}{3-t}\right| + c$$

b) $\int \frac{\cos x}{2-\cos x}\,dx = 2\int \frac{\left(\frac{1-t^2}{1+t^2}\right)}{\left\{2-\left(\frac{1-t^2}{1+t^2}\right)\right\}}\frac{dt}{(1+t^2)}$

$$= 2\int \frac{1-t^2}{2(1+t^2)^2-1+t^4}\,dt$$

$$= 2\int \frac{1-t^2}{(t^2+1)(3t^2+1)}\,dt$$

If we write $\frac{1-t^2}{(t^2+1)(3t^2+1)} = \frac{At+B}{t^2+1} + \frac{Ct+D}{3t^2+1}$,

then $1-t^2 = (At + B)(3t^2+1) + (Ct + D)(t^2+1)$

$\therefore$ $1 = B + D$ (constants)

$0 = A + C$ (coefficients of t)

$-1 = 3B + D$ (coefficients of t^2)

$0 = 3A + C$ (coefficients of t^3)

$\therefore$ $A = C = 0, B = -1, D = 2$

$$\therefore \text{Answer} = -2\int \frac{dt}{t^2+1} + 4\int \frac{dt}{3t^2+1}$$

$$= -2\tan^{-1}(t) + \frac{4}{\sqrt{3}}\tan^{-1}(\sqrt{3}\,t) + c$$

$$= -2\,\frac{x}{2} + \frac{4}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan\frac{x}{2}) + c$$

$$= -x + \frac{4}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan\frac{x}{2}) + c$$

E7) a) $\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx = \int \frac{t^2}{1+t}\,4t^3dt$ if $t = \sqrt[4]{x}$

$$= 4\int \frac{t^5}{1+t}\,dt$$

$$= 4\int\left[t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1}\right]dt$$

$$= 4\left[\frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right] + c$$

$$= 4\left[\frac{x^{5/4}}{5} - \frac{x}{4} + \frac{x^{3/4}}{3} - \frac{x^{1/2}}{2} + x^{1/4} - \ln|x^{1/4}+1|\right] + c$$

b) In $\int \frac{dx}{(2-x)\sqrt{1-2x+3x^2}}$ put $2-x = \frac{1}{t}$. Then $\frac{dx}{dt} = \frac{1}{t^2}$

$$\int \frac{dx}{(2-x)\sqrt{1-2x+3x^2}} = \int \frac{dx}{(2-x)\sqrt{3(2-x)^2 - 10(2-x) + 9}}$$

$$= \int \frac{t}{\sqrt{\frac{3}{t^2} - \frac{10}{t} + 9}}\,\frac{1}{t^2}\,dt$$

$$= \int \frac{dt}{\sqrt{9t^2 - 10t + 3}} = \int \frac{dt}{\sqrt{(3t - \frac{5}{3})^2 + \frac{2}{9}}}$$

$$= \frac{1}{3}\int \frac{dt}{\sqrt{(t - \frac{5}{9})^2 + (\frac{\sqrt{2}}{9})^2}} = \frac{1}{3}\sinh^{-1}\left(\frac{t - 5/9}{\sqrt{2}/9}\right) + c$$

$$= \frac{1}{3}\sinh^{-1}\frac{9}{\sqrt{2}}\left(\frac{1}{2-x} - \frac{5}{9}\right) + c$$

$$= \frac{1}{3}\sinh^{-1}\left(\frac{5x-1}{\sqrt{2}\,(2-x)}\right) + c$$

Solution of E8, Unit II

i) $\sin^{-1}\left(\frac{x}{3}\right) + c$

ii) $\cosh^{-1}\left(\frac{u}{2}\right) + c$

iii) Putting $2x = t$, we get $\frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\tan^{-1} t + c$

$= \frac{1}{2}\tan^{-1}(2x) + c$

iv) $\frac{1}{\sqrt{10}}\tan^{-1} x\sqrt{\frac{2}{5}} + c$

v) Put $x^2 = y$, then $\int \frac{x}{\sqrt{x^4-1}}\,dx = \frac{1}{2}\int \frac{dy}{\sqrt{y^2-1}} = \frac{1}{2}\cosh^{-1} y + c$

$= \frac{1}{2}\cosh^{-1}(x^2) + c$

vi) $\frac{1}{12}\tan^{-1}\left(\frac{t^3}{4}\right) + c$

vii) $\frac{1}{3}\sin^{-1}\left(\frac{u^3}{2}\right) + c$

viii) $\sin^{-1}(x-1) + c$ (as in Example 13).

ix) $\frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2}}$

Let $x + \frac{1}{2} = y$. Then the given integral is

$\sinh^{-1}\left(\frac{2y}{\sqrt{3}}\right) + c = \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

x) $\cosh^{-1}\left(\frac{y+3}{2}\right) + c$

xi) $x - \tan^{-1} x + c$

dπ

UTTAR PRADESH
RAJARSHI TANDON OPEN UNIVERSITY

UGMM - 01
Calculus

Block

4

APPLICATIONS OF CALCULUS

BLOCK 4 INTRODUCTION

In this block we shall look at some of the applications of the concepts studied in the earlier blocks. We have already studied some geometrical applications of derivatives in Block 2. Here, in Unit 14, we shall study some of their applications to the theory of functions. In particular, we shall establish some inequalities, and calculate the approximate values of some functions at some points.

We have mentioned in the introduction of Unit 3 that the problem of finding the area under a curve partially led to the invention of Calculus. In Unit 15 we shall see how the concept of a definite integral can be used to find the area under a curve. But area is not the only thing which can be calculated by using integration. You will see in Unit 16, that the concept of integration can be used to calculate even the lengths of some plane curves. After reading Unit 16 you will also realise the usefulness of a definite integral in finding the volumes and surface areas of solids of revolution.

In Block 3 we have seen various techniques of integration. But still, there are some functions, whose antiderivatives cannot be written in terms of any of the known functions, no matter which technique we use. In spite of this, we can find approximate values of the definite integrals of many of these functions by using numerical integration. We shall study two methods of numerical integration in Unit 15. These also come in handy if we want to find the definite integral of a function whose values are known only at some specific points.

With this block we come to the end of this course on calculus. In this course we have dealt with functions of one variable only. You can study the calculus of functions of several variables in the course titled **Advanced Calculus**.

UNIT 14 APPLICATIONS OF DIFFERENTIAL CALCULUS

Structure

14.1 INTRODUCTION

In this unit we see some immediate applications of the concepts which we have studied in Blocks 1 and 2.

Differential Calculus has varied applications. You have already seen some applications to geometrical, physical and practical problems in Units 8 and 9. In this unit, we shall study some applications to the theory of real functions.

There are some questions in mathematics, that can be asked even before knowing differentiation, but can be solved with ease by using the theorems on differentiation. In other words, differentiation is a useful tool in solving problems that arise independent of this notion, for example,

a) Is the function $\sin x + \cos x$ increasing in $[0, \pi/4]$?

b) Do we have the inequality $e^x > 1+x$ for all x?

c) What is the limit of $\frac{1-e^x}{\sin x}$ when x tends to 0?

d) What is the approximate value of $\cos 1°$?

Do you notice that these questions do not involve any concept that you have not studied earlier? You could have asked them earlier. You could have even answered some of them. Here we shall see how the theorems of Unit 7 can be systematically applied to yield solutions to such questions. Keep Block 2 ready with you since you will need to read the relevant portions from the units in that block.

Objectives

After studying this unit you should be able to:

- recognise the equivalence of some properties of functions (like monotonicity and positiveness or negativeness of its derivative),
- prove some inequalities using the mean value theorems,
- apply Taylor's series to obtain approximate values of certain functions at certain points.

14.2 MONOTONIC FUNCTIONS

In this section we employ differentiation to decide whether a given function is monotonic or not.

First we recall some terms from Unit 1. We say that a function from an interval I to R is monotonic if it is either an increasing function on I or a decreasing function on I.

It is said to be an increasing function on I, if $x \leq y$ implies $f(x) \leq f(y)$. Increasing functions may also be thought of as order-preserving functions.

Let us also recall that

- every constant function is an increasing function,
- the identity function is an increasing function,
- the function $f(x) = 2x + 3$ is an increasing function,
- the function $f(x) = \sin x$ is not an increasing function on R.

This is because even though $0 < 3\pi/2$, we have $\sin 0 = 0 > -1 = \sin 3\pi/2$. However, on the interval $[0, \pi/2]$, $\sin x$ is an increasing function.

Further, we know that a function f from I to R is said to be a decreasing function if $x \le y$ implies $f(x) \ge f(y)$. Decreasing functions are the order-reversing functions.

Here are some examples:

- The function $f(x) = 4-2x$ is decreasing.
- The function $f(x) = \sin x$ is decreasing in the interval $[\pi/2, \pi]$.

Do you agree that each constant function is both increasing and decreasing?

Warning: It is incorrect to say that if a function is not increasing, then it is decreasing. It may happen that a function is neither increasing nor decreasing. For instance, if we consider the interval $[0, \pi]$, the function $\sin x$ is neither increasing nor decreasing. It is increasing on $[0, \pi/2]$ and decreasing on $[\pi/2, \pi]$. There are other functions that are even worse. They are not monotonic on any sub-interval also. But most of the functions that we consider are not so bad.

Usually, by looking at the graph of the function one can say whether the function is increasing or decreasing or neither. The graph of an increasing function does not fall as we go from left to right, while the graph of a decreasing function does not rise as we go from left to right. But if we are not given the graph, how do we decide whether a given function is monotonic or not? Theorem 1 gives us a criterion to do just that.

Theorem 1: Let I be an open interval. Let $f : I \to R$ be differentiable. Then

a) f is increasing if and only if $f'(x) \ge 0$ for all x in I.

b) f is decreasing if and only if $f'(x) \le 0$ for all x in I.

Proof: a) Let f be increasing. Let $x \in I$. Then,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Since f is increasing, if $h > 0$, then $x+h > x$ and $f(x+h) \ge f(x)$.
Hence $f(x+h) - f(x) \ge 0$. If $h < 0$, $x+h < x$, and $f(x+h) \le f(x)$.
Hence $f(x+h) - f(x) \le 0$.
So either $f(x+h) - f(x)$ and h are both non-negative or they are both non-positive. Therefore,

$\dfrac{f(x+h) - f(x)}{h}$ is non-negative for all non-zero values of h.

Therefore, $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ must also be non-negative.

Thus, $f'(x) \ge 0$.

Conversely, let $f'(x) \ge 0$, for all x in I.
Let $a < b$ in I. We shall prove that $f(a) \le f(b)$. By mean value theorem (Theorem 3, Unit 7),

$\dfrac{f(b) - f(a)}{b - a} = f'(c)$ for some $c \in]a, b[\subset I$.

Since $f'(c) \ge 0$, we have

$\dfrac{f(b) - f(a)}{b - a} \ge 0$. Also $b - a > 0$. It follows that $f(b) - f(a) \ge 0$, or $f(b) \ge f(a)$.

Thus, $a < b$ implies $f(a) \le f(b)$. Therefore, f is increasing.

b) This can be proved similarly. We leave it as an exercise. It can also be deduced by applying part a) to the function, $-f$.

E E1) Prove Part b) of Theorem 1.

From the class of increasing functions we can separate out functions which are strictly increasing. The following definition gives the precise meaning of the term "strictly increasing function".

Definition 1: $f : I \to R$ is said to be strictly increasing if $a < b$ implies that $f(a) < f(b)$.

We can similarly say that a function defined on I is strictly decreasing if $a < b$ implies $f(a) > f(b)$. For example, a constant function is not strictly increasing, nor is it strictly decreasing. The function $f(x) = [x]$ too, is increasing, but not strictly increasing, whereas the function $f(x) = x$ is strictly increasing.

Fig. 1 shows the graphs of these three functions. In Fig. 1(a) the graph is horizontal. In Fig. 1(b) there are parts of the graph which are horizontal. But the graph in Fig. 1(c) has no horizontal portions, and rises as we go from left to right.

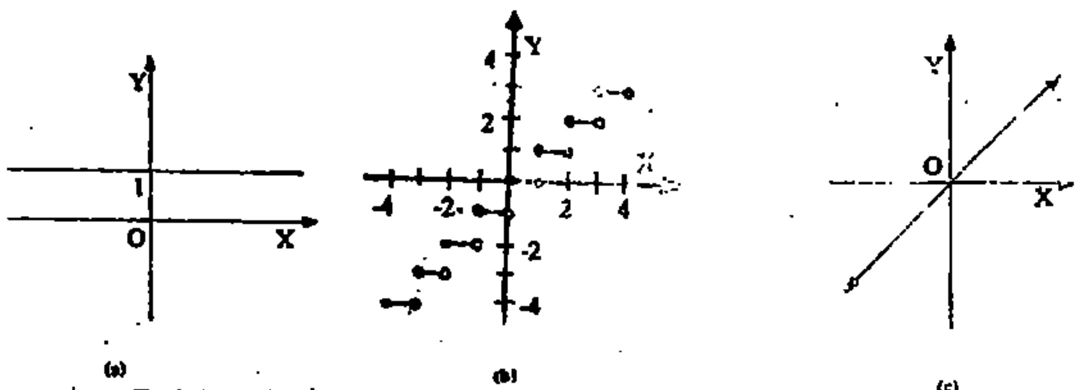


Fig. 1: (a) graph of $f(x) = 1$ (b) graph of $f(x) = [x]$ (c) graph of $f(x) = x$.

Now let us see whether strict monotonicity of a function is reflected by its derivative. We have the following theorem.

Theorem 2: a) Let f' be positive on I. Then f is strictly increasing on I.

b) Let f' be negative on I. Then f is strictly decreasing on I.

Proof: a) By Theorem 1, we know that since $f' > 0$ on I, f must be increasing on I. That is, $x < y \Rightarrow f(x) \leq f(y)$. We have only to prove that if $x < y$, then $f(x)$ cannot be equal to $f(y)$. Let, if possible, $f(x) = f(y)$, where $x < y$ in I.

Then, by Rolle's Theorem (Theorem 2, Unit 7) applied to the function f on the interval [x,y], we have f' (c)=0 for some x < c < y.

But this contradicts the assumption that f' is strictly positive on I. Hence f(x) cannot be equal to f(y) for any x<y in I. Thus, x < y ⇒ f(x) < f(y). That is, f is strictly increasing.

b) We indicate two different proofs for this part. One way is to imitate the proof of part a) by changing the symbol '<' to the symbol '>', wherever it occurs.

Another way is to consider −f, apply Theorem 2 a) to it and use the facts: $(-f)' = -f'$. Therefore f' is negative if and only if $(-f)'$ is positive. Also f is strictly decreasing if and only if −f is strictly increasing. Combining the results for increasing and decreasing functions we get the following corollary

Corollary 1: f is strictly monotonic on the interval I if f' is of the same sign throughout I.

You may have noticed that there is a difference between the statements of Theorem 1 and Theorem 2. We said:

"f is increasing if and only if f' is non-negative".

"If f' > 0, then f is strictly increasing".

It is natural to ask:

Can we have "if and only if" in Theorem 2 also? That is, if f is strictly increasing, does it follow that f' > 0? Unfortunately, we have a negative answer as shown in Example 2 below.

But before that, our first example shows how among many methods available to prove the monotonicity, the one using differentiation is the simplest (provided the function is differentiable, of course).

Example 1: Let $f(x) = x^3$ for all x in **R**. We shall prove that the function f is increasing.

First Method: Let x < y. We want to prove that $x^3 \le y^3$. Consider two cases.

Case 1: x and y are of the same sign. (Either both are positive or both are negative.) In this case xy > 0. Now,

$y^3 - x^3 = (y - x)(y^2 + xy + x^2) \ge 0$ since both $y - x$ and $y^2 + yx + x^2$ are non-negative.

Case 2: Let x and y be not of the same sign. Since x < y, this means that x < 0 < y. (Note that if either x or y is zero, it comes under Case 1.) Therefore, $x^3 < 0 < y^3$. Because, the cube of a negative number is negative, and the cube of a positive number is positive.

Thus, in both cases $x^3 \le y^3$. In fact, we have the strict inequality $(x^3 < y^3)$, indicating that $x \to x^3$ is a strictly increasing function.

Second Method: Let x < y. We want to prove that $x^3 \le y^3$.

Now $(y^3 - x^3) = (y - x)(y^2 + yx + x^2)$

Here $y - x > 0$ (since x < y). Also, $y^2 + yx + x^2 \ge 0$ because

$$y^2 + yx + x^2 = \frac{1}{2}[(y^2 + x^2) + (y^2 + 2yx + x^2)].$$

$$= \frac{1}{2}[y^2 + x^2 + (y + x)^2] \ge 0,$$

since the square of any number is non-negative. Thus $y^3 - x^3$ is a product of two non-negative numbers and, hence is non-negative.

Third Method (Using Differentiation): Let $f(x) = x^3$. Then $f'(x) = 3x^2$. This is always non-negative. Therefore using Theorem 1 we can say that f is an increasing function

Example 2: Here we give an example of a strictly increasing function whose derivative is not strictly positive.

Let $f: \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = x^3$ (see Fig. 2). It is strictly increasing because,

$x < y \Rightarrow y - x > 0$ and $x^2 + y^2 > 0$

$\Rightarrow y^3 - x^3 = (y - x)(y^2 + yx + x^2)$

$= \frac{1}{2}(y - x)[(x^2 + y^2) + (x + y)^2] > 0$

$\Rightarrow x^3 < y^3$.

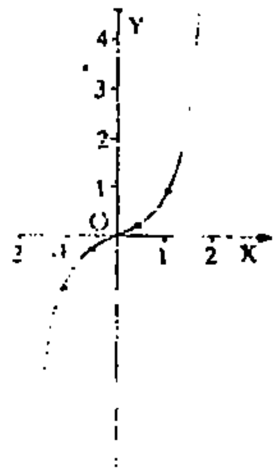


Fig. 2

Its derivative is not strictly positive because $f'(0) = 0$. Our next example describes two different ways in which non-monotonicity of a function can be proved.

Example 3: To prove that the function $f : x \to \sin x + \cos 2x$ is not monotonic on the interval $[0, \pi/4]$, we can proceed as follows:

First Method: We shall consider three points, 0, $\pi/6$ and $\pi/10$ belonging to $[0, \pi/4]$.

Then, $f(0) = \sin 0 + \cos 0 = 1$

$f(\pi/6) = \sin \frac{\pi}{6} + \cos \frac{\pi}{3} = \frac{1}{2} + \frac{1}{2} = 1$

$f(\pi/10) = \sin \frac{\pi}{10} + \cos \frac{\pi}{5} = 0.3090 + 0.8090 > 1$

We have $0 < \pi/10 < \pi/6$ and $f(0) < f(\pi/10) > f(\pi/6)$.

Therefore, f is neither increasing, nor decreasing on $[0, \pi/4]$. Or, We can say that f is not monotonic on $[0, \pi/4]$.

Second Method: (Using Differentiation)

Let $f(x) = \sin x + \cos 2x$

Then $f'(x) = \cos x - 2 \sin 2x$

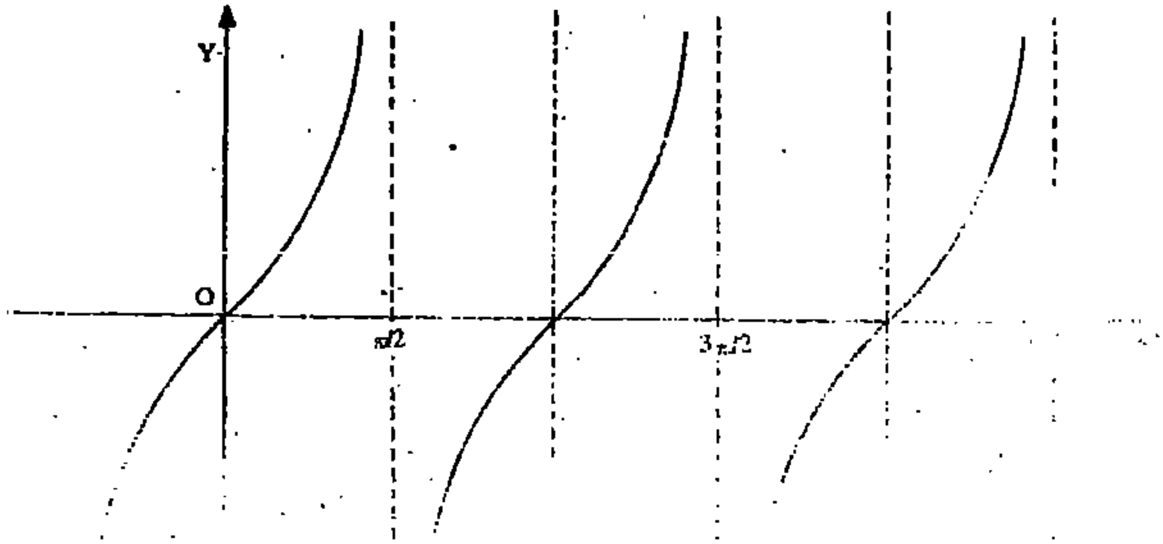
Now, $f'(0) = 1 - 0 = 1$ and $f'(\pi/4) = \frac{1}{\sqrt{2}} - 2 \times 1 < 0$.

Thus f' is of different signs at 0 and $\pi/4$.

Therefore f is not monotonic on $[0, \pi/4]$.

Our next example warns us in dealing with functions not defined at some points.

Example 4: If the function $f : x \to \tan x$ is defined on an interval, we can prove that it is an increasing function there. See Fig. 3.



Now consider the interval $[0, \pi]$. Can we prove that $\tan x$ is increasing on this interval? Suppose we argue as follows:

$f'(x) = \sec^2 x$, and $\sec^2 x \geq 0 \ \forall \ x$ since the square of any quantity is non-negative. Hence by Theorem 1, f is an increasing function on $[0, \pi]$.

But if we take two points $\frac{\pi}{4}$ and $\frac{2\pi}{3}$ in $[0, \pi]$,

then $f\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1$, and $f\left(\frac{2\pi}{3}\right) = \tan \frac{2\pi}{3} = -\sqrt{3}$.

Thus, $\frac{\pi}{4} < \frac{2\pi}{3}$, but $\tan \frac{\pi}{4} \not\leq \tan \frac{2\pi}{3}$.

This indicates that $\tan x$ is not an increasing function on $[0, \pi]$.
So where did we go wrong? We can explain it as follows:

In the interval $[0, \pi]$, there is a point, namely $\frac{\pi}{2}$, where tan is not defined. Hence its derivative does not exist at that point, and therefore we can not apply Theorem 1.

What we proved is that this function is increasing in an interval, provided it is defined throughout this interval. It is only when it is defined, that we can differentiate it and apply our theorems.

In the next example we use an additional property of continuous functions in the first method, and repeated differentiation in the second method.

Example 5: Let us prove that the function $x \to \sin x + \cos x$ is increasing on $[0, \pi/4]$ and decreasing on $[\pi/4, \pi/2]$.

First Method: Recall that for all x in the first quadrant (i.e. if $0 \leq x \leq \pi/2$), sin x, cos x and tan x are non-negative.

Let $f(x) = \sin x + \cos x$, then $f'(x) = \cos x - \sin x$
We note that $f'(0) = 1$,
$f'(\pi/4) = 0$,
$f'(\pi/2) = -1$.
$x = \pi/4$ is the only point in $[0, \pi/2]$ at which $f'(x) = 0$.
Because $f'(x) = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1$. But tan x is strictly increasing on $[0, \pi/2]$, by Example 4. So, it cannot take the value 1 at any point other than $\pi/4$.

Also, f' is a continuous function because the sine and cosine functions are continuous.
Therefore, f' cannot take negative values in $[0, \pi/4]$.

Explanation: If a continuous function takes a positive value at 0 and a negative value at some x, then it must be zero somewhere in between 0 and x. In this problem, the continuous function f' cannot take the value zero between 0 and x if $x < \pi/4$.

Therefore, since f' is non-negative on $[0, \pi/4]$, f is increasing on $[0, \pi/4]$.

Similarly, f' cannot take positive values in $[\pi/4, \pi/2]$, because its value at $\pi/2$ is negative. It follows that f is decreasing on $[\pi/4, \pi/2]$.

Second Method: Let $f(x) = \sin x + \cos x$, and $f'(x) = \cos x - \sin x$. To prove that f is increasing on $[0, \pi/4]$, we have to prove that f' is non-negative on $[0, \pi/4]$. We first note that $f'(0) = 1$ and $f'(\pi/4) = 0$. It is enough to prove that f' is decreasing on $[0, \pi/4]$ (for then, all values of f' in this interval will be between 0 and 1). For this purpose we consider $f''(x) = -\sin x - \cos x$. Also see Fig. 4(a) and (b).

We note that $f''(x) \leq 0$ for all x in the first quadrant, and in particular for all x in $[0, \pi/4]$. Therefore f' is decreasing on $[0, \pi/4]$. Therefore, (since $f'(0) = 1$ and $f'(\pi/4) = 0$), f' is non negative on $[0, \pi/4]$. Therefore f is increasing on $[0, \pi/4]$.

Next, we shall prove that f' is non-positive on $[\pi/4, \pi/2]$. First, we note that $f'(\pi/4) = 0$ and $f'(\pi/2) = -1$. Also, f' is decreasing on $[\pi/4, \pi/2]$, since $f''(x) \leq 0$ on $[\pi/4, \pi/2]$ Therefore all the values of f' on this interval are between 0 and -1 and hence cannot be positive. Therefore f is decreasing on $[\pi/4, \pi/2]$.

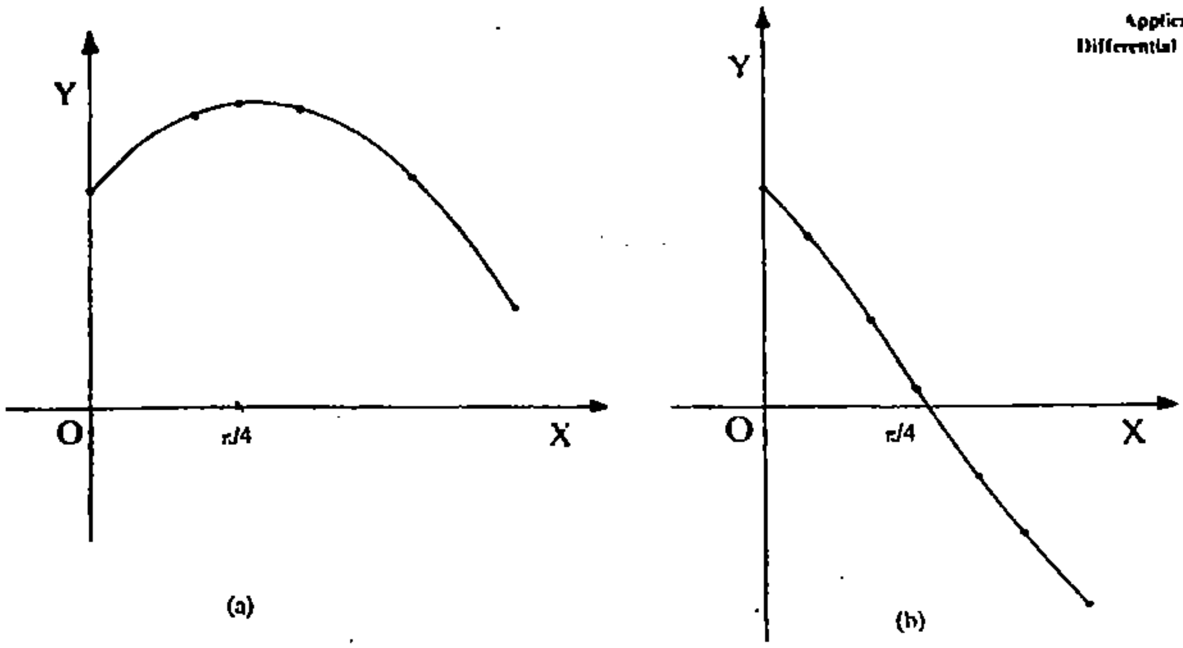


Fig. 4: (a) Graph of $f(x) = \sin x + \cos x$, (b) Graph of $f(x) = \cos x - \sin x$

If you have followed the arguments in these examples, you should not have any difficulty in solving these exercises.

E E 2) Assuming that e^x never takes negative values, prove that it is an increasing function on **R**.

E E 3) Prove that $\ln x$ is an increasing function on $]0, \infty[$.

E E 4) Using the fact that sin x and cos x are never negative in the first quadrant, prove by differentiation that sin x is an increasing function and cos x is a decreasing function on $[0, \pi/2]$.

E E 5) Which of the following functions are increasing on the interval given? Which of them are decreasing?

a) $x^2 - 1$ on $[0, 2]$

b) $2x^2 + 3x$ on $[-1/2, 1/2]$

c) e^x on $[0, 1]$

d) $x(x-1)(x+1)$ on $[-2, -1]$

e) $x \sin x$ on $[0, \pi/2]$

f) $\tan x + \cot x$ on $[0, \pi/4]$

E E 6) Prove that the following functions are not monotonic in the intervals given.

a) $2x^2+3x-5$ on $[-1, 0]$

b) $x(x-1)(x+1)$ on $[0, 2]$

c) $x \sin x$ on $[0, \pi]$

d) $\tan x + \cot x$ on $[0, \pi/2]$

E E 7) Give an example of a cubic polynomial that decreases on $]-\infty, 2]$, increases on $[2, 3]$ and again decreases on $[3, \infty[$.

(*Hint*: The derivative should change sign while passing through 2, and again while passing through 3.)

14.3 INEQUALITIES

Another application of differential calculus is to prove certain inequalities. In the three examples below we illustrate how some inequalities can be deduced from Taylor's series of Unit 6 and the mean value theorem (Theorem 3) of Unit 7, and from the theorems proved in Sec. 14.2.

Example 6: Suppose we want to prove $e^x \geq 1+x$ for all x in R.

Let $f(x)=e^x - x$, Then $f'(x)=e^x - 1$, $f''(x)=e^x$

We know that $e^x > 0$ for all x.

Therefore, $f'(x)$ is a strictly increasing function.

Also $f'(0)=0$.

Therefore there is no other point where f' vanishes.

Also, $f'(x) > 0$ if $x > 0$ $f'(x) < 0$ if $x < 0$.

Therefore f(x) is increasing on $]0, \infty[$.

So $x > 0$ implies $f(x) > f(0)$. This means $e^x - x > e^0 - 0 = 1$.

This proves $e^x > 1+x$, if $x > 0$.

It remains to prove this for negative values of x also. For this purpose we let $g(x)=e^x - xe^x$.

Then $g'(x)=e^x - (xe^x+e^x) = -xe^x < 0$ wherever $x > 0$.

$\Rightarrow$ g is strictly decreasing on $]0, \infty[$

$\Rightarrow g(x) < g(0)$ whenever $x > 0$.

$\Rightarrow e^x - xe^x < e^0 - 0.e^0 = 1$ for all $x > 0$.

In other words, $e^x < \frac{1}{1-x}$ for all $x > 0$.

Putting $y = -x$, we get $e^{-y} < \frac{1}{1+y}$, or

$e^y > 1+y$ for $y < 0$. In other words, $e^x > 1+x$ for all $x < 0$.

When $x=0$, $e^x=1=1+x$.

Thus, the inequality $e^x \geq 1+x$ is true for all values of x.

In the next example we give an inequality that is still better.

Example 7: We prove that $e^x > 1+x+\frac{x^2}{2}+\frac{x^3}{6}$ for all $x > 0$.

We have seen in Unit 6 that the Taylor's series expansion

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+ \ldots\ldots$ is valid for all x.

This proves the inequality

$e^x > 1+x+\frac{x^2}{2}+\frac{x^3}{6}$ whenever $x > 0$.

Fig. 5 represents the results of Examples 6 and 7.

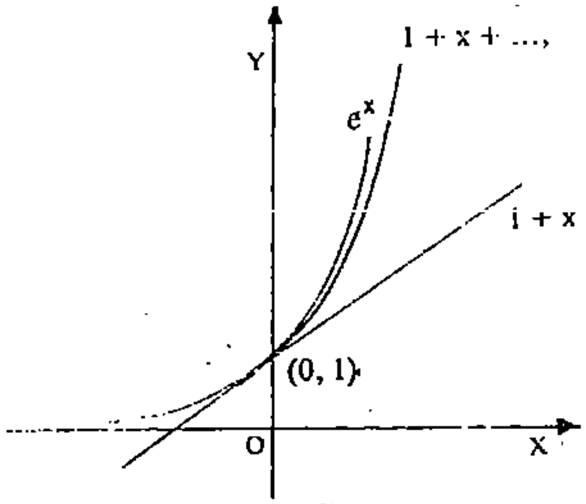


Fig. 5

Example 8: To prove $b^n - a^n < n b^{n-1}(b - a)$, wherever $0 < a < b$ and $n > 1$, we consider the function $f : x \to x^n$ on the interval $[a, b]$. It is continuous there. It is also differentiable in $]a, b[$. Therefore, by the mean value theorem, there is some c, $a < c < b$ such that

$$nc^{n-1} = f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cross-multiplying we get, $b^n - a^n = nc^{n-1}(b - a)$.

Therefore, it suffices to prove that $c^{n-1} < b^{n-1}$.

This is true because $0 < c < b$, and $n > 1$.

You can try these exercises now

E 8) Prove the following inequalities using the methods indicated alongside in brackets.

a) $\ln(1 + x) < x$ for all positive x
(first prove that $x - \ln(1 + x)$ is increasing).

b) $\tan^{-1} x < x$ for all positive x
(by mean value theorem on $[0, x]$ for $\tan^{-1} x$).

c) $e^x + e^{-x} > 2 \; \forall \; x$. (writing $e^x + e^{-x} - 2$ as a perfect square).

d) $e^x + e^{-x} > 2 \; \forall \; x$. (using the already proved result $e^x \geq 1 + x$).

e) $e^x - e^{-x} \geq 2x \; \forall \; x > 0$. (using the inequality in d) and differentiation).

14.4 APPROXIMATE VALUES

In the previous two sections we have seen how the concept of derivatives can be used in proving the monotonicity of a given differentiable function; and how this knowledge can be applied to prove some inequalities.

In this section we shall see how Taylor's series can be used to find approximate values of some functions at some points. We use the symbol $\simeq$ to mean approximately equal to.

Example 9: Taking the first two non-zero terms in Maclaurin's series for $\sin x$, we shall prove that $\sin 20^\circ$ is approximately equal to 0.342 (in symbols, $\sin 20^\circ \simeq 0.342$). Remember that in the formula

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$$

the angle x is always measured in radians. The same holds for cos x, tan x, and so on.

Now, $20^\circ = \frac{\pi}{9}$ radians. Therefore,

$$\sin \frac{\pi}{9} = \frac{\pi}{9} - \frac{(\pi/9)^3}{3!} + \ldots\ldots$$

$$\sin \frac{\pi}{9} \simeq \frac{\pi}{9} - \frac{(\pi/9)^3}{6}$$

Taking $\pi = 3.142$, we get this quantity to be

$$\frac{3.142}{9} - \frac{.0425}{6} \simeq 0.349 - 0.007 = 0.342.$$

If you look into a table of sines, you will find $\sin 20^\circ \simeq 0.342$. This shows that our approximation is really a good approximation. In fact, the tables are written by using precisely these methods.

Example 10: Let us find the approximate value of $(0.99)^{5/2}$ by taking three terms of Maclaurin's series for $(1 - x)^{5/2}$

Maclaurin's series for $(1 - x)^{5/2}$ is

$$1 - \frac{5}{2}x + \frac{(5/2)(3/2)}{2}x^2 + \ldots\ldots$$

We can write $(0.99)^{5/2}$ as $(1 - 0.01)^{5/2}$.

So when $x = 0.01$, taking the first three terms of Maclaurin's series, we get

$$(1 - 0.01)^{5/2} \simeq 1 - \frac{5}{2}(0.01) + \frac{15}{8}(0.0001)$$

That is, $(0.99)^{5/2} \simeq 0.975$.

The error is the difference between the actual value and the approximate value.

Example 11: We know that $\cos \frac{\pi}{6} = \sqrt{3}/2$. If the first two non-zero terms of Maclaurin's series for cos x are taken to approximate it, let us calculate the error, rounded off to two decimal places.

Maclaurin's series for $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$$

If we take the first two terms alone and put $x = \pi/6$, then

$$\cos \frac{\pi}{6} \simeq 1 - \frac{(\pi/6)^2}{2} \simeq 1 - \frac{1}{2}\left(\frac{3.142}{6}\right)^2$$

$$\simeq 1 - 0.274/2 = 0.863$$

The actual value is $\cos \frac{\pi}{6} = \sqrt{3}/2$. We know that $\sqrt{3}/2 = 0.866$ when rounded off to three decimal places.

We have found that $1 - \frac{(\pi/6)^2}{2} \simeq 0.863$ when rounded off to three decimal places

The error is 0.866 − 0.863 = 0.003.

When rounded off to two decimal places, the error is 0.00. (This means that the error is so negligible that there is no error at all when rounded off to two decimal places.)

See if you can find the approximate values in the following exercises.

E E 9) Find the approximate value of sin 31° by taking the first two non-zero terms of its Maclaurin's series.

E E 10) If the first three non-zero terms of Maclaurin's series for cos x are used to approximate $\cos \frac{\pi}{2}$, show that the error is less than 1/50.

E E 11) Find the value of cos 59°, rounded off to one decimal place

E 12) Find $(1.01)^{1/2}$ upto two decimal places.

That brings us to the end of this unit. Let us summarise what we have studied in it.

14.5 SUMMARY

In this unit we have studied the following results.

1)

If f is	then f' is
increasing	non-negative
decreasing	non-positive
constant	identically zero
monotonic	of same sign throughout

2)

If f' is	then f is
non-negative	increasing
non-positive	decreasing
(strictly) positive	strictly increasing
(strictly) negative	strictly decreasing
identically zero	constant
of same sign throughout	monotonic

3) Differentiation can be used

- to test whether a function is monotonic or not,
- to prove some inequalities,
- and to find some approximate values.

14.6 SOLUTIONS AND ANSWERS

E 1) Let f be a decreasing function. If $x \in I$, then $f'(x)$ exists, and

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$h > 0 \Rightarrow f(x+h) \leq f(x) \Rightarrow f(x+h) - f(x) \leq 0$$

$$h < 0 \Rightarrow f(x+h) \geq f(x) \Rightarrow f(x+h) - f(x) \geq 0.$$

So, $\frac{f(x+h) - f(x)}{h} \leq 0 \ \forall h \neq 0$

Hence $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \leq 0$.

Now, let $f'(x) \leq 0$ in I, and $a, b \in I$ s.t. $a < b$. Then $\exists\, c \in I$ s.t.

$$\frac{f(b) - f(a)}{b - a} = f'(c) \leq 0.$$

$b - a > 0 \Rightarrow f(b) - f(a) \leq 0$ or $f(b) \leq f(a)$.

$\Rightarrow$ f is a decreasing function.

E 2) Let $f(x)=e^x$. $f'(x)=e^x \geq 0 \ \forall x \in \mathbf{R}$.

$\Rightarrow f(x)=e^x$ is an increasing function on $\mathbf{R}$ (by Theorem 1).

E 3) If $f(x)=\ln x$, $f'(x)=\frac{1}{x}$. Now $\frac{1}{x} > 0 \ \ \forall x \in]0, \infty[$.

Hence $\ln x$ is an increasing function on $]0, \infty[$.

E 4) If $f(x)=\sin x$, then $f'(x)=\cos x$.

$\cos x \geq 0 \ \forall x \in [0, \pi/2]$.

Hence $\sin x$ is an increasing function on $[0, \pi/2]$. Similarly for $\cos x$.

E5) a) $f'(x)=2x \geq 0$ on $[0, 2]$: f is increasing on $[0, 2]$.

b) $f'(x)=4x+3 \geq 0$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$: f is increasing on $\left[-\frac{1}{2}, \frac{1}{2}\right]$

c) $f'(x)= -e^{-x} \leq 0$ on $[0, 1] \Rightarrow f$ is decreasing on $[0, 1]$.

d) increasing

e) increasing

f) $f'(x)=\sec^2 x - \mathrm{cosec}^2 x = \frac{-\cos 2x}{\sin^2 x \cos^2 x} \leq 0$ since $\cos 2x \geq 0$

$\forall x \in [0, \pi/4]$. $\therefore$ f is decreasing.

E 6) a) $f'(x)=4x+3$

$f'(x) \leq 0$ if $x \in [-1, -3/4]$ and $f'(x) \geq 0$ if $x \in [-3/4, 0]$.

b) $f'(x)=3x^2-1$. $f'(x) \leq 0$ if $x \in \left[0, \frac{1}{\sqrt{3}}\right]$

and $f'(x) \geq 0$ if $x \in \left[\frac{1}{\sqrt{3}}, 2\right]$

c) $f'(x)=\sin x + x\cos x \begin{cases} \geq 0 \text{ for } x \in [0, \pi/2] \\ \leq 0 \text{ for } x \in [3\pi/4, \pi]. \end{cases}$

d) similar argument

E 7) Let $f(x)=ax^3+bx^2+cx+d$

$f'(x)=3ax^2+2bx+c$

f has extreme at $x=2$ and $x=3$.

$f'(2)=0 \Rightarrow 12a+4b+c=0$

and $f'(3)=0 \Rightarrow 27a+6b+c=0$

$\therefore 15a+2b=0$

or $b=\frac{-15a}{2}$. $\therefore c=18a$

Suppose $a=-2$ and $d=0$.

Then $f(x)=-2x^3+15x^2-36x$ satisfies the given conditions.

E 8) a) $f(x)=x-\ln(1+x)$. $f'(x)=1-\frac{1}{1+x} \geq 0$ for all $x > 0$

$x-\ln(1+x)$ is increasing on $]0, \infty[$

$\Rightarrow \ln(1+x) < x \ \ \forall x \in]0, \infty[$.

b) $f(x)=\tan^{-1} x$.

Let $x > 0$. By the mean value theorem $\exists\, y \in]0, x[$ s.t.

$$\frac{\tan^{-1} x - \tan^{-1} 0}{x-0} = f'(y)$$

or $\frac{\tan^{-1} x}{x} = \frac{1}{1+y^2} < 1$ for $y \in]0, x[$.

$\therefore \tan^{-1} x < x \ \ \forall x > 0$.

c) $(e^x + e^{-x} - 2) = (e^{x/2} + e^{-x/2})^2 > 0 \quad \forall\, x.$

d) $e^x \geq 1 + x$ and $e^{-x} = e^y \geq 1 + y = 1 - x$ (if $y = -x$)
$\therefore e^x + e^{-x} \geq 1 + x + 1 - x = 2.$

e) Let $f(x) = e^x - e^{-x} - 2x$. Then $f'(x) = e^x + e^{-x} - 2 \geq 0.$
$\Rightarrow$ f is an increasing function
$\Rightarrow f(x) \geq f(0)$ for all $x > 0$
$\Rightarrow e^x - e^{-x} - 2x \geq 0$ for all $x > 0$.
$\Rightarrow e^x - e^{-x} \geq 2x$ for all $x > 0$.

E 9) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$

$$31^\circ = \frac{31\pi}{180} \text{ radians.}$$

$$\therefore \sin \frac{31\pi}{180} \simeq \frac{31\pi}{180} - \left(\frac{31\pi}{189}\right)^3 \cdot \frac{1}{6}.$$

$$\simeq \frac{31 \times 3.142}{180} - \left(\frac{31 \times 3.142}{180}\right)^3 \cdot \frac{1}{6}$$

$$= .5411 - (.5411)^3 \frac{1}{6}$$

$$= .5411 - .0264$$

$$= .5147.$$

E 10) If $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$
$f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$

$$f(x) = \cos x \simeq f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0)$$

$$\cos \frac{\pi}{2} = \cos 0 + \frac{\pi}{2} \times 0 - \left(\frac{\pi}{2}\right)^2 \frac{1}{2!} \times 1 + \left(\frac{\pi}{2}\right)^3 \times \frac{1}{3!} \times 0 + \left(\frac{\pi}{2}\right)^4 \frac{1}{4!} \times 1$$

$$= 1 - \frac{\pi^2}{8} + \frac{\pi^4}{16 \times 24}$$

$$\simeq 1 - \frac{(3.142)^2}{8} + \frac{(3.142)^4}{16 \times 24}$$

$$\simeq 1 - 1.234 + 0.235$$

$$= 0.001,$$

We know that $\cos \frac{\pi}{2} = 0$

$$\therefore \text{Error} = 0.001 < \frac{1}{50}$$

E 11) $\text{Cos } \frac{59\pi}{180} \simeq 1 - \left(\frac{59\pi}{180}\right)^2 \frac{1}{2!} \times 1$

$$= 1 - \left(\frac{3.142 \times 59}{180}\right)^2 \frac{1}{2!}$$

$$\simeq 1 - 0.530$$

$$= 0.470$$

$$\simeq 0.5$$

E 12) $(1 + x)^r = 1 + rx + \frac{r(r-1)}{2!} x^2 + \ldots$

$$\therefore (1.01)^{1/2} = (1 + 0.01)^{1/2} \simeq 1 + \frac{1}{2}(0.01) + \frac{\frac{1}{2} \cdot \frac{-1}{2}}{2!} (0.01)^2$$

$$\simeq 1 + 0.005 - 0.00001 \simeq 1.00$$

UNIT 15 AREA UNDER A CURVE

Structure

15.1 INTRODUCTION

When we introduced you to integration, we mentioned that the origin of the method of integration lies in the attempt to estimate the areas of regions bounded by plane curves. In this unit we shall see how to calculate the area under a given curve, when the equation of the curve is given in the Cartesian or polar or parametric form. This process is also called **quadrature**. We shall also study two methods of numerical integration. These are helpful when the antiderivative of the integrand cannot be expressed in terms of known functions, and the given definite integral cannot be exactly evaluated.

Objectives

After reading this unit you should be able to:

- use your knowledge of integration to find the area under a given curve whose equation is given in the Cartesian or polar or parametric form,
- recognise the role of numerical integration in solving some practical problems when some values of the function are known, but the function, as a whole, is not known,
- use trapezoidal and Simpson's rules to find approximate values of some definite integrals,
- compare the two rules of numerical integration.

15.2 AREA UNDER A CURVE

In this section we shall see how the area under a curve can be calculated when the equation of the curve is given in the

i) Cartesian form
ii) polar form
iii) parametric form.

Some curves may have a simple equation in one form, but complicated ones in others. So, once we have considered all these forms, we can choose an appropriate form for a given curve, and then integrate it accordingly. Let us consider these forms of equations one by one.

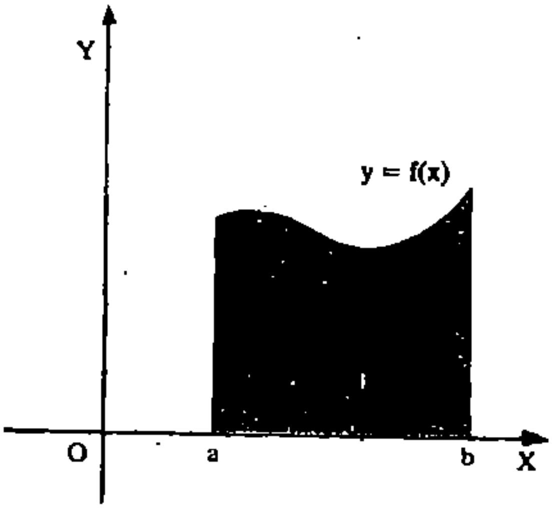
15.2.1 Cartesian Equation

We shall quickly recall what we studied in Sec. 2 of Unit 10. Let $y = f(x)$ define a continuous function of x on the closed interval [a, b]. For simplicity, we make the assumption that f(x) is positive for $x \in [a, b]$. Let R be the plane region in Fig. 1 (a) bounded by the graphs of the four equations:

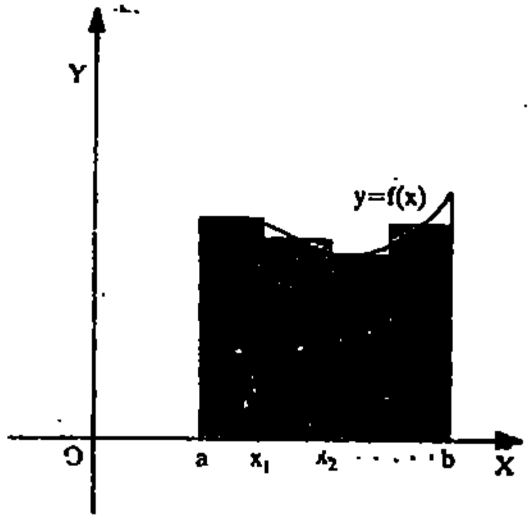
$y = f(x),\ y = 0,\ x = a,\ x = b.$

We divide the region R into n thin strips by lines perpendicular to the x-axis through the end points $x = a$ and $x = b$, and through many intermediate points which we indicate by $x_1, x_2, \ldots, x_{n-1}$. Such a subdivision, as you have already seen in Sec. 2 of Unit 10, is referred to as a **partition P_n** of the interval $[a, b]$ and is indicated briefly by writing

$$P_n = \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$$



(a)



(b)

Fig. 1

We write

$\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n$,

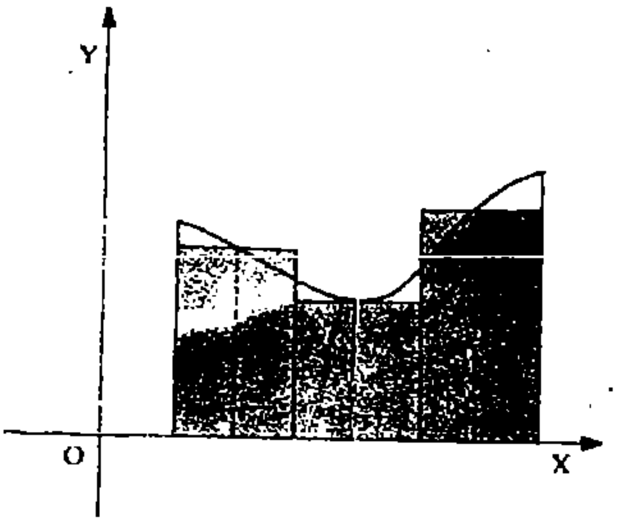
and take the set of n points

$T_n = \{t_1, t_2, \ldots, t_{n-1}, t_n\}$,

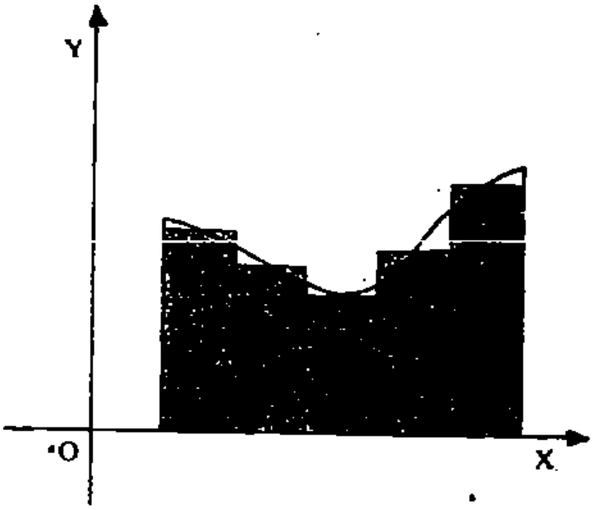
such that $x_{i-1} \leq t_i \leq x_i$ for $i = 1, 2, \ldots, n$. We now construct the n rectangles (Fig. 1 (b)) whose bases are the n sub-intervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$ induced by the partition P_n, and whose altitudes are $f(t_1), f(t_2), \ldots, f(t_i), \ldots, f(t_{n-1}), f(t_n)$. The

sum $\sum_{i=1}^{n} f(t_i) \; \Delta x_i$

of the areas of these n rectangles will be an approximation to the "area of R". Notice (Fig. 2(a) and (b)) that if we increase the number of sub-intervals, and decrease the length of each sub-interval, we obtain a closer approximation to the "area of R"



(a)



(b)

Fig. 2

Thus, we have

Definition 1: Let f be a real valued function continuous on [a, b], and let $f(x) \geq 0 \ \forall x \in [a, b]$. If the limit of $\sum_{i=1}^{n} f(t_i) \ \Delta x_i$ exists as the lengths of the sub-intervals, $\Delta x_i \to 0$, then that limit is the area A of the region R.

That is, $A = \lim_{\substack{\Delta x_i \to 0 \\ i=1,2,\ldots,n}} \sum_{i=1}^{n} f(t_i) \ \Delta x_i$

Compare this definition with that of a definite integral given in Unit 10. Over there we had seen that the definite integral,

$\int_a^b f(x)\,dx$ is the common limit of $\sum_{i=1}^{n} m_i \Delta x_i$ and $\sum_{i=1}^{n} M_i \Delta x_i$ as the Δx_i's $\to 0$.

Now since $m_i \leq f(t_i) \leq M_i \ \forall i$, we have

$$\sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} f(t_i) \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i$$

Hence if the limit of each of these as Δx_i's $\to 0$ exists, then by the Sandwich Theorem in Unit 2,

$$\lim_{\substack{\Delta x_i \to 0 \\ i=1,2,\ldots,n}} \sum_{i=1}^{n} m_i \ \Delta x_i \leq \lim_{\substack{\Delta x_i \to 0 \\ i=1,2,\ldots,n}} \sum_{i=1}^{n} f(t_i) \ \Delta x_i \leq \lim_{\substack{\Delta x_i \to 0 \\ i=1,2,\ldots,n}} \sum_{i=1}^{n} M_i \Delta x_i$$

Now, if $\int_a^b f(x)\,dx$ exists, then the first and the third limits here are equal, and therefore we get $A = \int_a^b f(x)\,dx$. ....(1)

The equality in (1) is a consequence of the definitions of the area of R and the definite integral $\int_a^b f(x)\,dx$. Since f(x) is assumed to be continuous on the interval [a, b], the integral in (1) exists, and hence yields the area of the region R under consideration.

From the Interval Union Property (Sec. 3, Unit 10) of definite integrals, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a \leq c \leq b. \qquad \text{....(2)}$$

This means if A_a^c, A_c^b, A_a^b denote the areas under the graph of $y = f(x)$ above the x-axis from a to c, from c to b and from a to b, respectively, (Fig. 3) then, if c is in between a and b, then we have

$$A_a^c + A_c^b = A_a^b \qquad \text{....(3)}$$

If we define $A_a^a = 0$, $A_b^b = 0$, then the above equation is true for $c = a$ and $c = b$ too.

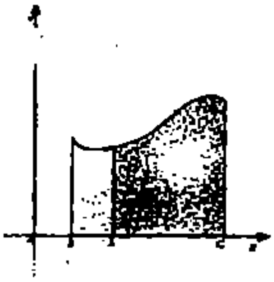


Fig. 3

Till now, we have assumed the function f(x) to be positive in the interval [a, b]. In general, a function f(x) may assume both positive and negative values in the interval [a, b]. To cover such a case, we introduce the convention about *signed* areas.

The area is taken to be positive above the x-axis as we go from left to right, and negative if we go from right to left. The function f(x) may be defined beyond the interval [a, b] also. In that case (3) is true even if c is beyond b, since according to our convention of signed areas, A_c^b will turn out to be a negative quantity (Fig. 4).

Thus, $A_a^b = A_a^c + A_c^b = A_a^c - A_b^c$.

Or, $A_a^b + A_b^c = A_a^c$.

Now, if $f(x) \le 0$ for all x in some interval [a, b], then by applying the definition of "area of R" to the function $-f(x)$, we get the area $A = -\int_a^b f(x)\,dx$.

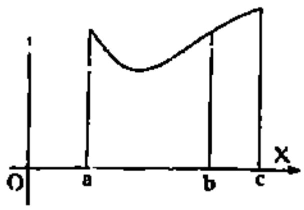


Fig. 4

If we do not take the negative sign, the value of the area will come out to be negative, since f(x) is negative for all $x \in [a, b]$. To avoid a "negative" area, we follow this convention. Thus, if $f(x) \le 0$ for $x \in [a, b]$ (Fig. 5), then the area between the ordinates x = a and x = b will be

$$A = -\int_a^b f(x)\,dx$$

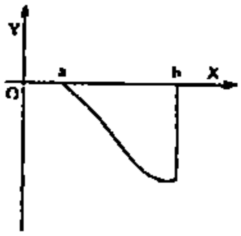


Fig. 5

The following examples will illustrate how our knowledge of evaluating definite integrals can be used to calculate certain areas.

Example 1: Suppose we want to find the area of the region bounded by the curve $y = 16 - x^2$, the x-axis and the ordinates x = 3, x = −3. The region R, whose area is to be found, is shown in Fig. 6.

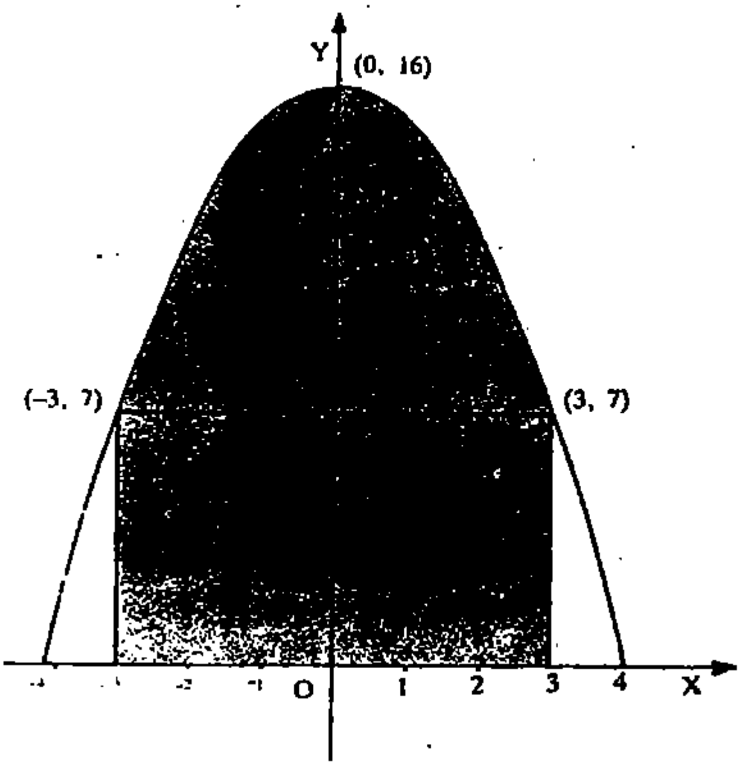


Fig. 6

The area A of the region R is given by

$$A = \int_{-3}^{3} (16 - x^2)\,dx$$

$$= \left[16x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= 78$$

Example 2: Consider the shaded region R in Fig. 7.

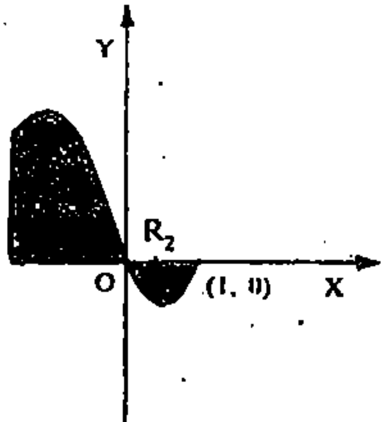


Fig. 7

R is composed of two parts, the region R_1 and the region R_2. We have
Area R = Area R_1 + Area R_2

The region R_1 is bounded above the x-axis by the graph of
$y = x^3 + x^2 - 2x$, $x = -2$ and $x = 0$.

Hence,

$$\text{Area } R_1 = \int_{-2}^{0} (x^3 + x^2 - 2x)\, dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_{-2}^{0}$$

$$= \frac{8}{3}$$

The region R_2 is bounded below the x-axis by the graph of

$y = x^3 + x^2 - 2x$, $x = 0$ and $x = 1$.

Hence,

$$\text{Area } R_2 = -\int_{0}^{1} (x^3 + x^2 - 2x)\, dx$$

$$= -\left[\frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_{0}^{1} = \frac{5}{12}$$

Therefore, Area $R = \frac{8}{3} + \frac{5}{12} = \frac{37}{12}$.

In this example we had to calculate area R_1 and area R_2 separately, since the region R_2 was below the x-axis. Therefore, according to our convention

$$\text{Area } R_2 = -\int_{0}^{1} f(x)\, dx.$$

If we calculate $\int_{-2}^{1} f(x)\, dx$, it will amount to calaulating

$\int_{-2}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx = \text{area } R_1 - \text{area } R_2$, which would be a wrong estimate of area R.

Example 3: Let us find the area of the smaller region lying above the x-axis and included between the circle $x^2+y^2=2x$ and the parabola $y^2=x$.

On solving the equations $x^2+y^2=2x$ and $y^2=x$ simultaneously, we get (0, 0), (1, 1), (1, -1) as the points of intersection of the given curves. We have to find th area of the region R bounded by OAPBO (Fig. 8).

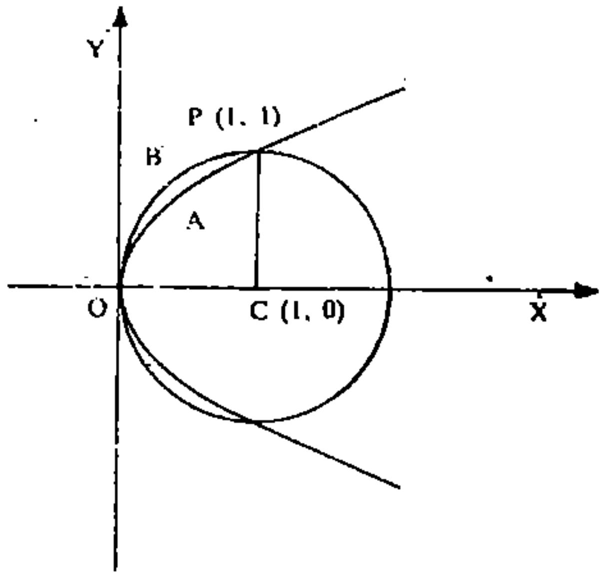


Fig. 8

From the figure we see that area of region OAPBO

= area of region OCPBO – area of region OCPAO

$$= \int_0^1 \sqrt{2x - x^2}\, dx - \int_0^1 \sqrt{x}\, dx$$

Now, $\int_0^1 \sqrt{2x - x^2}\, dx = \int_0^1 \sqrt{1 - (1 - x)^2}\, dx$

$$= \int_{\pi/2}^{0} \cos\theta\,(-\cos\theta)\, d\theta, \text{ on putting } 1 - x = \sin\theta$$

$$= \int_{\pi/2}^{0} -\cos^2\theta\, d\theta = \int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{4}$$

Also, $\int_0^1 \sqrt{x}\, dx = \frac{2}{3}$.

Therefore, the required area $= \left(\frac{\pi}{4} - \frac{2}{3}\right)$

Try to solve these exercises now.

E E 1) Find the area under the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

E 2) Find the area bounded by the x-axis, the curve $y = e^x$, and the ordinates $x = 1$ and $x = 2$.

E 3) Find the area of the region bounded by the curve $y = 5x - x^2$, $x = 0$, $x = 5$ and lying above the x-axis.

E 4) Find the area cut off from the parabola $y^2 = 4ax$ by its latus rectum, $x = a$

E 5) Find the area between the parabola $y^2=4ax$ and the chord $y=mx$.

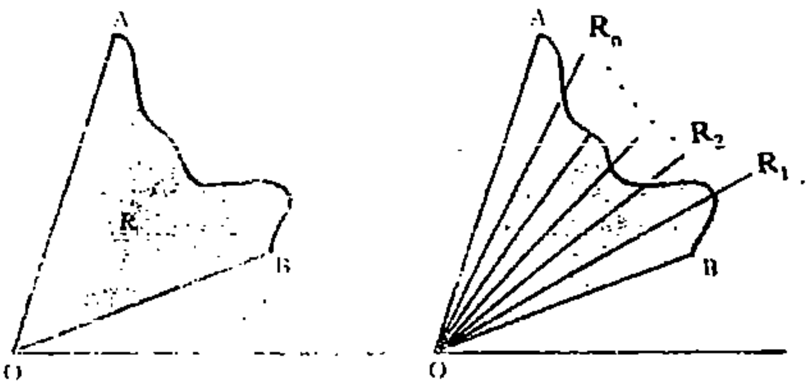
In this sub-section we have derived a formula (Formula (1)) to find the area under a curve when the equation of the curve is given in the cartesian form. With slight modifications we can use this formula to find the area when the curve is described by a pair of parametric equations.

We shall take a look at curves given by parametric equations a little later. But first, let us consider the curves given by a polar equation.

15.2.2 Polar Equations

Sometimes the Cartesian equation of a curve is very complicated, but its polar equation is not so. Cardioids and spirals which you have encountered in Unit 9 are examples of such curves. For these curves it is much simpler to work with their polar equation rather than with the Cartesian ones. In this sub-section we shall see how to find the area under a curve if the equation of the curve is given in the polar form. Here we shall try to approximate the given area through the areas of a series of circular sectors. These circular sectors will perform the same function here as rectangles did in Cartesian coordinates.

Let $r=f(\theta)$ determine a continuous curve between the rays $\theta=\alpha$ and $\theta=\beta$ $(\beta-\alpha \leq 2\pi)$. We want to find the area A(R) of the shaded region R in Fig. 9 (a).



(a) (b)

Imagine that the angle AOB is divided into n equal parts $\Delta\theta$.

Then $\Delta\theta = \frac{\beta - \alpha}{n}$. This amounts to slicing R into n smaller regions, $R_1, R_2, \ldots, R_n$, as shown in Fig. 9(b).
Then clearly

$$A(R) = A(R_1) + A(R_2) + \ldots + A(R_n),$$

$$= \sum_{i=1}^{n} A(R_i).$$

Now let us take the i^{th} slice R_i, and try to approximate its area. Look at Fig. 10.

Fig. 10

The area of a sector of a circle of radius r and sectorial angle $\Delta\theta$ is $\frac{1}{2} r^2 \Delta\theta$.

Suppose f attains its minimum and maximum values on $[\theta_{i-1}, \theta_i]$ at u_i and v_i.

Then $\frac{1}{2}[f(u_i)]^2 \Delta\theta \le A(R_i) \le \frac{1}{2}[f(v_i)]^2 \Delta\theta$.
From this we get

$$\sum_{i=1}^{n} \frac{1}{2}[f(u_i)]^2 \Delta\theta \le \sum_{i=1}^{n} A(R_i) \le \sum_{i=1}^{n} \frac{1}{2}[f(v_i)]^2 \Delta\theta$$

The first and the third sums in this inequality are the lower and upper Riemann sums (ref. Unit 10) for the same definite integral,

namely, $\int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 \, d\theta$.

Therefore, by applying the sandwich theorem as $\Delta\theta \to 0$, we get

$$A(R) = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_\alpha^\beta r^2 \, d\theta \qquad \ldots(4)$$

We shall illustrate the use of this formula through some examples. Study them carefully, so that you can do the exercises that follow later.

Example 4: Suppose we want to find the area enclosed by the cardioid $r = a(1 - \cos\theta)$.

We have $r = 0$ for $\theta = 0$ and $r = 2a$ for $\theta = \pi$.

Since $\cos\theta = \cos(-\theta)$, the cardioid is symmetrical about the initial lines AOX (Fig. 11).

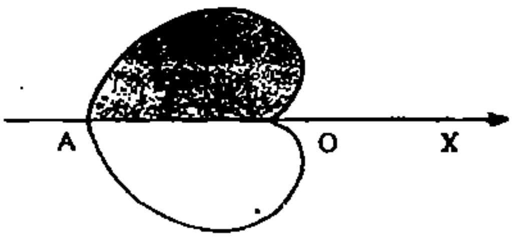


Fig. 11

Hence the required area A, which is twice the area of the shaded region in Fig. 11, is given by

$$A = 2\int_0^\pi \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^\pi a^2 (1 - \cos\theta)^2 \, d\theta$$

$$= 4a^2 \int_0^\pi \sin^4 \frac{\theta}{2} \, d\theta, \text{ since } \cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= 8a^2 \int_0^{\pi/2} \sin^4 \phi \, d\phi, \text{ where } \phi = \frac{\theta}{2}$$

$$= 8a^2 \frac{3}{4} \frac{1}{2} \frac{\pi}{2}$$ by applying the reduction formula from Section 3 of Unit 12.

$$= \frac{3}{2} a^2 \pi.$$

In the case of some Cartesian equations of higher degree it is often convenient to change the equation into polar form. The following example gives one such situation.

Example 5: To find the area of the loop of the curve

$x^5 + y^5 = 5ax^2y^2$,

we change the given equation into a polar equation by the transformation $x = r\cos\theta$ and $y = r\sin\theta$. Thus, we obtain

$$r = \frac{5a\cos^2\theta \sin^2\theta}{\cos^5\theta + \sin^5\theta},$$

which yields $r = 0$ for $\theta = 0$ and $\theta = \pi/2$. Hence, area A of the loop is that of a sectorial area bounded by the curve and radius vectors $\theta = 0$ and $\theta = \pi/2$, that is, the area swept out by the radius vector as it moves from $\theta = 0$ to $\theta = \pi/2$. See Fig. 12.

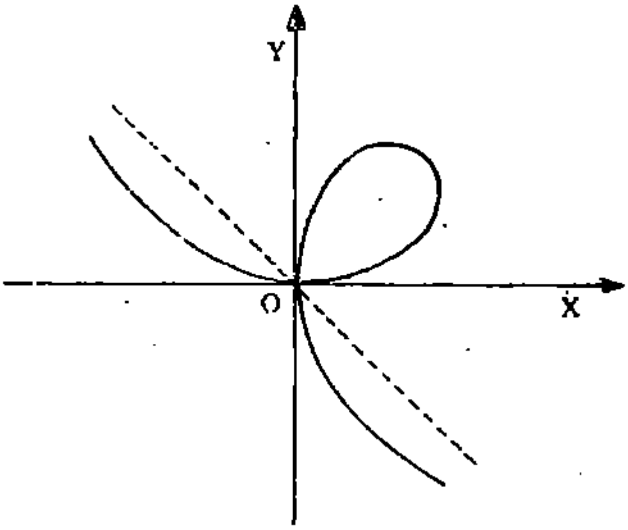


Fig. 12

Thus,

$$A = \frac{1}{2} \int_0^{\pi/2} \frac{25a^2 \cos^4\theta \sin^4\theta}{(\cos^5\theta + \sin^5\theta)^2} \, d\theta$$

$$= \frac{25}{2} a^2 \int_0^{\pi/2} \frac{\tan^4\theta \sec^2\theta}{(1 + \tan^5\theta)^2} \, d\theta$$

$$= \frac{5}{2} a^2 \int_1^{\infty} \frac{dt}{t^2}, \text{ where } t = 1 + \tan^5\theta.$$

$$= \frac{5}{2} a^2 [-1/t]_1^{\infty} = \frac{5}{2} a^2$$

Try to do these exercises now.

E E 6) Find the area of a loop of the curve $r = a \sin 3\theta$.

E E 7) Find the area enclosed by the curve $r = a \cos 2\theta$ and the radius vectors $\theta = 0$, $\theta = \pi/2$.

E E 8) Find the area of the region outside the circle $r = 2$ and inside the lemniscate $r^2 = 8 \cos 2\theta$.

[*Hint:* First find the points of intersection. Then the required area = the area under the lemniscate − the area under the circle.]

15.2.3 Area Bounded by a Closed Curve

Now we shall turn our attention to closed curves whose equations are given in the parametric form.
Let the parametric equations

Here we shall consider curves which do not intersect themselves.

$x = \phi(t),\ y = \psi(t),\ t \in [\alpha, \beta]$,

where $\phi(\alpha) = \phi(\beta)$ and $\psi(\alpha) = \psi(\beta)$, represent a plane closed curve (Fig. 13).

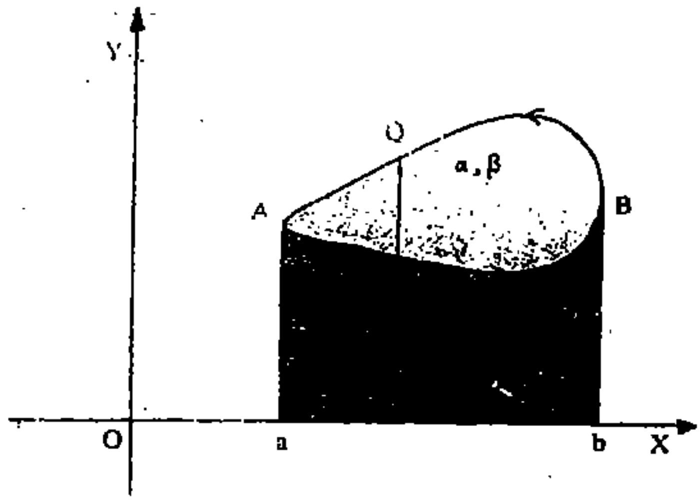


Fig. 13

This means that as the parameter t increases from a value α to a value β, the point P(x, y) describes the curve completely in the counter clockwise sense. Since the curve is closed, the point on it corresponding to the value β is the same as the point corresponding to the value α. This is reflected by the conditions $\phi(\alpha) = \phi(\beta)$ and $\psi(\alpha) = \psi(\beta)$.

Suppose further that the curve is cut at most in two points by every line drawn parallel to the x or y-axis. We also assume that the functions ϕ and ψ are differentiable, and that the derivatives ϕ' and ψ' do not vanish simultaneously. Let the point R on the closed curve correspond to the values α and β, i.e., at R we have $\phi(\alpha) = \phi(\beta)$ and $\psi(\alpha) = \psi(\beta)$.

Now suppose A is a point on the curve which has the least x-coordinate, a. Similarly, suppose B is a point on the curve which has the greatest x-coordinate, b. Thus the lines $x = a$ and $x = b$ touch the curve in points A and B, respectively. Further let t_1 and t_2 be the values of t that correspond to A and B, respectively. Then

$\alpha < t_2 < t_1 < \beta$.

Let a point Q correspond to $t = t_3$ such that $t_2 < t_3 < t_1$. The area of the region enclosed is $S = S_2 - S_1$, where S_2 and S_1 are the areas under the arcs AQB and ARB, respectively (see Fig. 13). Hence,

$$S_2 = \underset{(AQB)}{\int_a^b} y\,dx \text{ and } S_1 = \underset{(ARB)}{\int_a^b} y\,dx$$

Now, as a point P(x, y) moves from B to A along BQA, the value of the parameter increases from t_2 to t_1. Therefore,

$$\underset{(BQA)}{\int_b^a} y\,dx = \int_{t_2}^{t_1} y\,\frac{dx}{dt}\,dt$$

Note how we have modified formula (1) for a pair of parametric equations.

Hence $S_2 = -\int_{t_2}^{t_1} y\,\frac{dx}{dt}\,dt$

Now the movement of P from A to B along ARB, can be viewed in two parts: From A to R and from R to B. As P moves from A to R, the value of the parameter increases from t_1 to β, and as P moves from R to B, t increases from α to t_2.

Remember the point R corresponds to $t = \alpha$ and also to $t = \beta$.

$$\text{Therefore } S_1 = -\int_{a\,(ARB)}^{b} y\,dx = \int_{t_1}^{\beta} y\frac{dx}{dt}\,dt + \int_{\alpha}^{t_2} y\frac{dx}{dt}\,dt$$

Thus, we have

$$S = \int_{a\,(AQB)}^{b} ydx - \int_{a\,(ARB)}^{b} ydx = S_2 - S_1$$

$$= -\int_{t_2}^{t_1} y\frac{dx}{dt}\,dt - \int_{t_1}^{\beta} y\frac{dx}{dt}\,dt - \int_{\alpha}^{t_2} y\frac{dx}{dt}\,dt = -\int_{\alpha}^{\beta} y\frac{dx}{dt}\,dt \qquad \text{....(i)}$$

Note that the negative sign is due to the direction in which we go round the curve as marked by arrows in Fig. 13.

Similarly, by drawing tangents to the curve that are parallel to the x-axis, it can be shown that

$$S = \int_{\alpha}^{\beta} x\frac{dy}{dt}\,dt \qquad \text{....(ii)}$$

From (i) and (ii), we get

$$2S = \int_{\alpha}^{\beta}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt.$$

Hence, the area enclosed is

$$S = \frac{1}{2}\int_{\alpha}^{\beta}(xdy - ydx) \qquad \text{....(5)}$$

We can use any of the formulas (i), (ii) and (5) above for calculating S. But in many cases you will find that formula (5) is more convenient because of its symmetry.

Example 6: Let us find the area of the astroid

$x = a\cos^3 t,\ y = b\sin^3 t,\ 0 \le t \le 2\pi.$

The region bounded by the astroid is shown in Fig. 14.

The area A of the region is given by

$$A = \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_0^{2\pi}\{a\cos^3 t\,(3b\sin^2 t\cos t) - b\sin^3 t\,(-3a\cos^2 t\sin t)\}\,dt$$

$$= \frac{3ab}{2}\int_0^{2\pi}\cos^2 t\sin^2 t\,dt$$

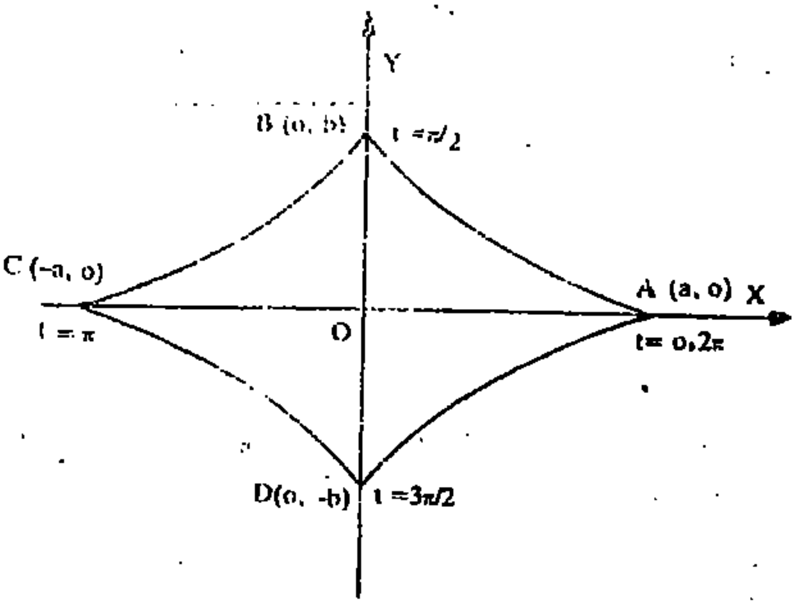


Fig. 14

We have seen in Section 3 of Unit 11 that

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$$

$$= 2\int_0^{a} f(x)\,dx, \quad \text{if } f(2a - x) = f(x).$$

Here $\cos^2(2\pi - t)\sin^2(2\pi - t) = \cos^2 t \sin^2 t$.

Hence $\int_0^{2\pi} \cos^2 t \sin^2 t\,dt = 2\int_0^{\pi} \cos^2 t \sin^2 t\,dt$.

Therefore, $A = 3ab \int_0^{\pi} \cos^2 t \sin^2 t\,dt$.

Now, by a similar argument we can say that

$$A = 6\,ab \int_0^{\pi/2} \cos^2 t \sin^4 t\,dt,$$

$$= \frac{3\pi\,ab}{8}, \text{ by using the reduction formula from Section 4 of Unit 12.}$$

You can solve these exercises now.

E 9) Find the area of the curve
$x = a\,(3\sin\theta - \sin^3\theta)$, $y = a\cos^3\theta$, $0 \le \theta \le 2\pi$.

E 10) Find the area enclosed by the curve

$x = a \cos \theta + b \sin \theta + c$

$y = a' \cos \theta + b' \sin \theta + c'$, $0 \le \theta \le 2\pi$

E 11) Find the area of one of the loops of the curve $x = a \sin 2t$, $y = a \sin t$.

(*Hint:* first find two values of t which give the same values of x and y, and take these as the limits of integration.)

15.3 NUMERICAL INTEGRATION

In many practical problems, the values of the integrand, that is, the function whose integral is required, are known only at some chosen points. For example,

x	x_0	x_1	x_2	x_3	x_4	x_5
$y = f(x)$	y_0	y_1	y_2	y_3	y_4	y_5

In some cases, no simple integral is known for the given integrand. For example, the function $\frac{\sin x}{x}$ does not have a simple indefinite integral. In such a situation the integral $\int_a^b f(x)dx$ cannot be evaluated exactly. But we can find an approximate value of the integral by considering the sum of the areas of inscribed (inner) or circumscribed (outer) rectangles, as we have seen at the beginning of this unit. In this section, we shall describe two more methods for approximating the value of an integral:

i) Trapezoidal Rule and ii) Simpson's Rule.

15.3.1 Trapezoidal Rule

We know that a given definite integral can be approximated by inner and outer rectangles. A better method is given by the trapezoidal rule in which we approximate the area of each strip by the area of a trapezium. Such an approximation is also called a **linear** approximation since the portions of the curve in each strip are approximated by line segments. As before, we divide the interval [a, b] into n subintervals, each of length

$\Delta x = \frac{(b-a)}{n}$, by using the points $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x, \ldots,$

$x_{n-1} = a + (n-1)\Delta x$ between $x_0 = a$ and $x_n = b$. Then

$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \ldots + \int_{x_{n-1}}^{b} f(x)\,dx$$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\,dx.$$

Now, we approximate the first integral on the RHS by the area of the trapezium $aP_0P_1x_1$ (Fig. 15), the second by $x_1P_1P_2x_2$ and so on, thus getting

The area of a trapezium = ½ (the sum of its parallel sides) × height.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(y_0 + y_1)\,\Delta x + \frac{1}{2}(y_1 + y_2)\,\Delta x + \ldots + \frac{1}{2}(y_{n-1} + y_n)\,\Delta x.$$

$$= \left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)\Delta x \qquad \ldots(6)$$

where $y_0 = f(x_0)$, $y_1 = f(x_1), \ldots, y_n = f(x_n)$

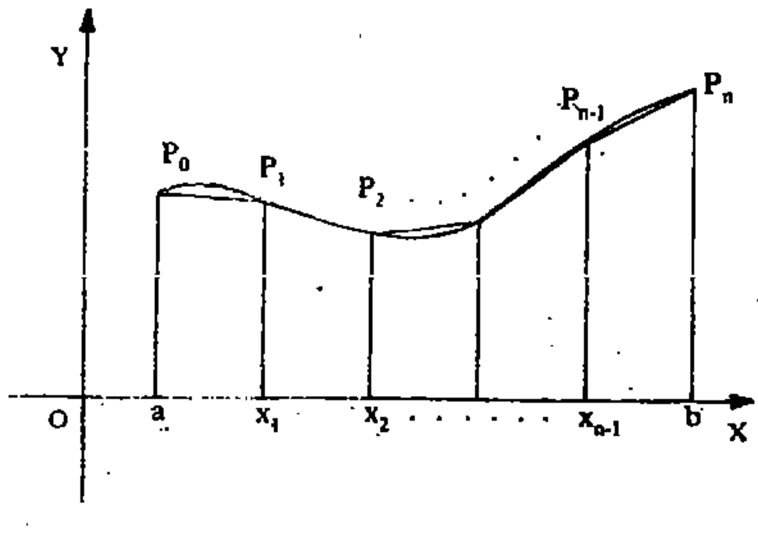


Fig. 15

The formula (6) is known as the trapezoidal rule for approximating the value of a definite integral.

See if you can solve this exercise now.

E 12) Use the trapezoidal rule to estimate the following integrals with the given value of n.

a) $\int_1^2 x^2\,dx,\ n=4$ b) $\int_1^4 \sqrt{x^2+4}\,dx,\ n=6.$

15.3.2 Simpson's Rule

In this method, instead of approximating the given curve by line segments, we approximate it by segments of a simple curve such as a parabola.

The area under the arc of the parabola

$y = Ax^2 + Bx + C$

between $x = -h$ and $x = h$ (Fig. 16) is given by

$$A_1 = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx$$

$$= \frac{2Ah^3}{3} + 2Ch$$

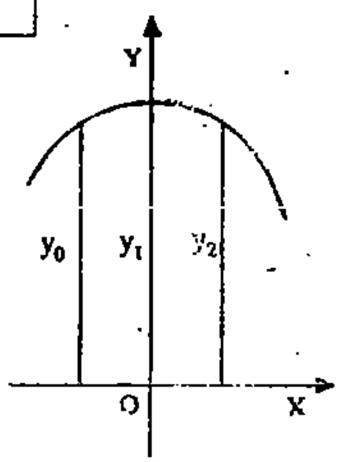


Fig. 16

Since the curve passes through the points, $(-h, y_0)$, $(0, y_1)$ and (h, y_2), substituting the coordinates of these points in the equation of the parabola, we get

$$y_0 = Ah^2 - Bh + C$$
$$y_1 = C$$
$$y_2 = Ah^2 + Bh + C,$$

from which we get

$$C = y_1$$
$$Ah^2 - Bh = y_0 - y_1$$
$$Ah^2 + Bh = y_2 - y_1$$
$$\Rightarrow 2\,Ah^2 = y_0 + y_2 - 2y_1$$

Hence, on substitution we get

$$A_1 = \frac{h}{3}\,[2Ah^2 + 6C] = \frac{h}{3}\,[(y_0 + y_2 - 2y_1) + 6y_1]$$

or $A_1 = \dfrac{h}{3}\,[y_0 + 4y_1 + y_2]$ ....(7)

To obtain Simpson's rule, we apply the above result to successive pieces of the curve $y = f(x)$ between $x = a$ and $x = b$. For this we divide $[a, b]$ into n sub-intervals, each of width h.

Then we approximate the portion of the curve in each pair of sub-intervals by an arc of a parabola passing through the end points of that portion of the curve and the point corresponding to the common point (Fig. 17) of those sub-intervals.

Now consider the first two sub-intervals $[a, x_1]$ and $[x_1, x_2]$. The points on the curve corresponding to a, x_1 and x_2 are P_0, P_1 and P_2, respectively. Let us draw a parabola passing through P_0, P_1 and P_2. We will assume that the portion of the curve passing through P_0, P_1 and P_2 coincides with this parabola. See Fig. 17.

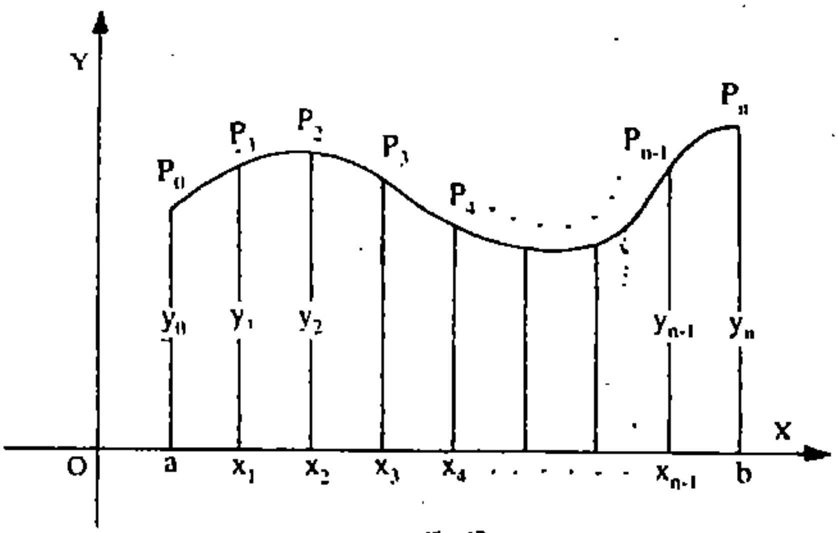


Fig. 17

Similarly we can approximate the portion of the curve in the interval $[x_2, x_4]$ by a parabola passing through the points P_2, P_3 and P_4. We shall repeat this process for the remaining pairs of intervals. Now the area under the parabola $P_0\,P_1\,P_2$ is given by

$A_1 = \dfrac{h}{3}\,(y_0 + 4y_1 + y_2)$. (On using formula (7)).

Similarly, the area under the parabola passing through the points P_2, P_3, P_4 is given by

$$A_3 = \frac{h}{3}\,[y_2 + 4y_3 + y_4].$$

Next, we use formula (7) for the parabola passing through the points P_4, P_5, ... and get the area

$A_5 = \dfrac{h}{3}\,[y_4 + 4y_5 + y_6]$, and so on.

Note, that to approximate the whole area under the given curve in this manner, the number n of the subdivisions of the interval [a, b] must be even. Summing all the areas we obtain

$$A = \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n]$$

as the total area. The above formula can also be written in the form

$$A = \frac{h}{3}[y_0 + y_{2m} + 2(y_2 + y_4 + \ldots + y_{2m-2}) + 4(y_1 + y_3 + \ldots + y_{2m-1})],$$

where we have taken n = 2m. Note that $y_0, y_1, y_2, \ldots, y_{2m}$ are the values of the function, f, for $x = a, a+h, a+2h, \ldots, a+2mh = b$, respectively. Hence by Simpson's rule, we have

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_{2m}) + 2(y_2 + y_4 + \ldots + y_{2m-2}) + 4(y_1 + y_3 + \ldots + y_{2m-1})].$$

The following example will help us compare the accuracy of the trapezoidal and Simpson's rules. You will find that Simpson's rule is more accurate than the trapezoidal rule.

Example 7: Taking four subdivisions of the interval [1, 3], let us find the approximate value of $\int_1^3 x^2 dx$ by the trapezoidal rule and also by Simpson's rule.

Division by 4 gives 0.5 as the width of each sub-interval of [1, 3]. The values of the integrand at these points of subdivision are given in the following table:

x	1	1.5	2	2.5	3
y = f (x)	1 (y_0)	2.25 (y_1)	4 (y_2)	6.25 (y_3)	9 (y_4)

Using the formula for the trapezoidal rule, we obtain

$$\int_1^3 x^2\,dx = \left(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \frac{1}{2}y_4\right)\Delta x$$

$$= \left(\frac{1}{2} + 2.25 + 4 + 6.25 + \frac{9}{2}\right)0.5 = 8.75 \quad \ldots(a)$$

Using the formula for Simpson's rule, we get

$$\int_1^3 x^2\,dx = \frac{0.5}{3}[1 + 9 + 2(4) + 4(2.25 + 6.25)]$$

$$= \frac{1}{6}[10 + 8 + 34] = 8.66 \ldots \text{ or } 8.67 \quad \ldots(b)$$

The actual value of the definite integral

$$\int_1^3 x^2\,dx = x^3/3 = 8.66 \ldots = 8.67 \quad \ldots(c)$$

On comparing (a) with the actual value in (c), we observe that the value given by the trapezoidal rule has an error of + 0.08. Comparison of (b) with (c) shows that the error in the value given by Simpson's rule is zero in this case.

Example 8: Let us use the trapezoidal rule with six subdivisions to evaluate

$\int_0^\pi \sqrt{\sin x}\,dx$. We shall also find the value of the integral by using Simpson's rule.

We have

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$y=\sqrt{\sin x}$	0	$\sqrt{1/2}$	$\sqrt{\sqrt{3}/2}$	1	$\sqrt{\sqrt{3}/2}$	$\sqrt{1/2}$	0

Using trapezoidal rule, we obtain

$$\int_0^{\pi} \sqrt{\sin x}\, dx = \left[\frac{1}{2}\times 0 + \sqrt{1/2} + \sqrt{\sqrt{3}/2} + 1 + \sqrt{\sqrt{3}/2} + \sqrt{1/2} + \frac{1}{2}\times 0\right]\frac{\pi}{6}$$

$$= \left(\sqrt{2} + 1 + 2\sqrt{\sqrt{3}/2}\right)\frac{\pi}{6} = 2.23$$

Using Simpson's rule, we have

$$\int_0^{\pi} \sqrt{\sin x}\, dx \simeq \left(\frac{1}{3}\times\frac{\pi}{6}\right)[(0+0)+2(\sqrt{\sqrt{3}/2}+\sqrt{\sqrt{3}/2})+4(\sqrt{1/2}+1+\sqrt{1/2})]$$

$$= \frac{4\pi}{18}(\sqrt{\sqrt{3}/2}+1+\sqrt{2}) \simeq 2.33$$

Example 9: A river is 80 m wide. The depth d in meters at a distance x m from one bank is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Let us find, approximately, the area of cross-section. Applying Simpson's rule with $n=8$, we obtain the area of cross-section

$$A = \frac{10}{3}[(0+3)+2(7+12+14)+4(4+9+15+8)]$$

$$= \frac{10}{3}[3+66+144] = 710 \text{ sq.m.}$$

As we have seen in this example, Simpson's rule is very useful in approximating the area of irregular figures like the cross-sections of lakes and rivers.

See if you can do these exercises now.

E E.13) a) Use Simpson's rule to evaluate the following, taking the given value of n.

$$\int_0^{\pi} \frac{\sin x}{x}\, dx,\ n = 4$$

b) From the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2},$$ calculate π, using Simpson's rule with $h=0.1$.

E 14) A curve is drawn through the points (1, 2), (1.5, [illegible] .8), (3, 3), (3.5, 2.6) and (4, 2.1). Estimate the area betv[illegible] x-axis and the ordinate $x = 1$, $x = 4$.

E 15) A river has width 30 meters. If the depth y meters at distance x meters from one bank be given by the table,

x	0	5	10	15	20	25	30
y	0	1.2	2.1	2.4	1.6	0.6	0

Find the approximate area of cross-section.

E 16) The velocity of a train, which starts from rest, is given by the following table, the time being reckoned in minutes from the start and the speed in kms/hr.

Min	2	4	6	8	10	12	14	16	18	20
Kms/hr	10	18	25	29	32	20	11	5	2	0

Estimate approximately the total distance run in 20 minutes.

Now let us quickly recall what we have done in this unit.

15.4 SUMMARY

In this unit we have covered the following points:

1) The knowledge of integration is helpful in finding areas enclosed by plane curves when their equations are known in

   a) Cartesian form: $A = \int_a^b y \, dx$

   b) Polar form: $A = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta$.

2) The area bounded by a closed curve given by parametric equations is

$$A = \int_\alpha^\beta y \frac{dx}{dt} = \frac{1}{2} \int_\alpha^\beta \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) dt.$$

3) When the integral cannot be exactly evaluated, we can use the method of numerical integration. The two methods given here are:

   a) Trapezoidal rule:

$$\int_a^b f(x) \, dx = \left(\frac{1}{2} y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2} y_n \right) \Delta x,$$

   where n is the number of sub-divisions of [a, b], and Δx the length of each sub-interval.

   b) Simpson's rule:

$$\int_a^b f(x) \, dx = \frac{h}{3} [(y_0 + y_{2m}) + 2(y_2 + y_4 + \ldots + y_{2m-2}) + 4(y_1 + y_3 + \ldots + y_{2m-1})],$$

   when [a, b] is divided into 2m sub-intervals of length h.

15.5 SOLUTIONS AND ANSWERS

E1) $\int_0^\pi \sin x \, dx = -\cos x]_0^\pi$

$$= -\cos \pi + \cos 0 = 1 + 1 = 2.$$

E2) $\int_1^2 e^x \, dx = e^x]_1^2 = e^2 - e.$

E3) $\int_0^5 (5x - x^2) \, dx = \left[\frac{5x^2}{2} - \frac{x^3}{3} \right]_0^5 = \frac{125}{6}$

E4) $y = 2\sqrt{ax}$

$$A = 2 \int_0^a 2 \sqrt{ax} \, dx = 4 \sqrt{a} \left. \frac{x^{3/2}}{3/2} \right]_0^a = 4 \sqrt{a} \frac{a^{3/2}}{3/2} = \frac{8a^2}{3}.$$

E5) Points of intersection of $y^2 = 4ax$ and $y = mx$ are $(0,0)$ and $(4a/m^2, 4a/m)$.

$$\therefore A = \int_0^{4a/m^2} 2 \sqrt{ax} \, dx - \int_0^{4a/m^2} mx \, dx = 2 \sqrt{a} \left. \frac{x^{3/2}}{3/2} \right]_0^{4a/m^2} - \left. \frac{mx^2}{2} \right]_0^{4a/m^2}$$

$$= \frac{4\sqrt{a}}{3} \left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2} \left(\frac{4a}{m^2}\right)^2 = \frac{8a^2}{3m^3}.$$

E6) For a loop, we should find two distinct, consecutive values of θ, for which we get the same value of r.

If $r_1 = a \sin 3\theta_1$, and $r_2 = a \sin 3\theta_2$,

$r_1 = r_2 \Rightarrow \sin 3\theta_1 - \sin 3\theta_2 = 0.$

or $2 \cos \frac{3(\theta_1 + \theta_2)}{2} \sin \frac{3(\theta_1 - \theta_2)}{2} = 0$

$\therefore \frac{3(\theta_1 + \theta_2)}{2} = \frac{\pi}{2}$ or $\frac{3(\theta_1 - \theta_2)}{2} = 0$

$\therefore \theta_1 + \theta_2 = \frac{\pi}{3}$ or $\theta_1 = \theta_2$.

Thus, $\theta_1 = 0$ and $\theta_2 = \pi/3$ will give the same value of r.

$$\therefore A = \frac{1}{2} \int_0^{\pi/3} r^2 \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

$$= \frac{a^2}{6} \int_0^{\pi} \sin^2 u \, du \quad \text{if } u = 3\theta.$$

$$= \frac{a^2\pi}{12}.$$

E7) $A = \frac{1}{2} \int_0^{\pi/2} a^2 \cos^2 2\theta \, d\theta$

$$= \frac{a^2}{4} \int_0^{\pi} \cos^2 \phi \, d\phi, \; \phi = 2\theta$$

$$= \frac{a^2}{4} \left(\int_0^{\pi/2} \cos^2 \phi \, d\phi + \int_0^{\pi/2} \cos^2 (\pi - \phi) \, d\phi \right)$$

$$= \frac{a^2}{4}\left(\int_0^{\pi/2}\cos^2\phi\, d\phi + \int_0^{\pi/2}\cos^2\phi\, d\phi\right)$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\cos^2\phi\, d\phi = \pi a^2/8.$$

E8) Points of intersection are given by

$8\cos 2\theta = 4$ or $\cos 2\theta = 1/2$, i.e., $\cos^2\theta = 3/4$

$\therefore\ \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Because of symmetry w.r.t. the initial line, the required area A = 2 (the area under the lemniscate above the initial line from $\theta = 0$ to $\theta = \pi/6$ and from $\theta = 5\pi/6$ to $\theta = \pi$ minus the area under the wide from $\theta = 0$ to $\theta = \pi/6$ and from $\theta = 5\pi/6$ to $\theta = \pi$)

$$= 2\left[\frac{1}{2}\int_0^{\pi/6} 64\cos^2 2\theta\, d\theta + \frac{1}{2}\int_{5\pi/6}^{\pi} 64\cos^2 2\theta\, d\theta - \frac{1}{2}\int_{5\pi/6}^{\pi} 4\, d\theta\right.$$

$$= 8\sqrt{3} + 28\pi/3$$

E9) $x = a(3\sin\theta - \sin^3\theta),\ y = a\cos^3\theta$

$$\frac{dx}{d\theta} = 3a(\cos\theta - \sin^2\theta\cos\theta) = 3a\cos^3\theta$$

$$\therefore \int_0^{2\pi} y\frac{dx}{d\theta}\, d\theta = 3a^2\int_0^{2\pi}\cos^6\theta\, d\theta = 12a^2\int_0^{\pi/2}\cos^6\theta\, d\theta$$

$$= 12a^2\,\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \text{ (reduction formula)}$$

$$= \frac{15a^2\pi}{8}$$

E10) $\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta,\ \frac{dy}{d\theta} = -a'\sin\theta + b'\cos\theta$

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = (ab' - ba') + (ac' - ca')\sin\theta + (cb' - bc')\cos\theta$$

$$\therefore \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta = (ab' - ba')\pi$$

E11) A loop of this curve lies between $t = 0$ and $t = \pi$

$$\int_0^{\pi} x\frac{dy}{dt} = a^2\int_0^{\pi}\sin 2t\cos t\, dt = 2a^2\int_0^{\pi}\sin t\cos^2 t\, dt$$

$$= 4a^2\int_0^{\pi/2}\sin t\cos^2 t\, dt$$

$$= 4a^2\left.\frac{\cos^3 t}{3}\right|_0^{\pi/2} = \frac{4a^2}{3}$$

E12) a)

x	0	1/2	1	3/2	2
x^2	0	1/4	1	9/4	4

$$A = \left(\frac{1}{2}\times 0 + 1/4 + 1 + 9/4 + \frac{1}{2}\times 4\right)\frac{1}{2}$$
$$= 2.75.$$

b)

x	1	1.5	2	2.5	3	3.5	4
$\sqrt{x^2+4}$	$\sqrt{5}$	2.5	$2\sqrt{2}$	$\sqrt{10.25}$	$\sqrt{13}$	$\sqrt{16.25}$	$2\sqrt{5}$

$$A = \left(\frac{1}{2}\times\sqrt{5} + 2.5 + 2\sqrt{2} + \sqrt{10.25} + \sqrt{13} + \sqrt{16.25} + \sqrt{5}\right)\frac{1}{2}$$
$$= 9.76.$$

E13) a)

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\frac{\sin x}{x}$	1	$4/\sqrt{2}\pi$	$2/\pi$	$4/\sqrt{2}\pi$	0

$$\therefore \int_0^{\pi}\frac{\sin x}{x}\,dx = \frac{\pi}{12}\left[1+0+2(2/\pi)+4(4/\sqrt{2}\pi+4/\sqrt{2}\pi)\right]$$
$$= \frac{\pi}{12}\left[1+\frac{4+16\sqrt{2}}{\pi}\right]$$
$$= \frac{\pi}{12}+\frac{1+4\sqrt{2}}{3}$$
$$= 1.0665.$$

b)

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\frac{1}{1+x^2}$	1	.99	.96	.92	.86	.8	.73	.67	.61	.55	.5

$$\therefore \int_0^1 \frac{1}{1+x^2}\,dx = \frac{.1}{3}\left[\frac{3}{2}+2(.96+.86+.73+.61)\right.$$
$$\left.+\ 4(.99+.92+.8+.67+.55)\right]$$
$$= 0.7846$$

Now $\pi/4 = 0.7846 \Rightarrow \pi = 3.1384$.

E14)

1	1.5	2	2.5	3	3.5	4
2	2.4	2.27	2.8	3	2.6	2.1

$$A = 0.5(1+2.4+2.8+2.27+3+2.6+1.05)$$
$$= 0.5(15.12) = 7.56 \text{ (trapezoidal rule).}$$

E15) $A = \frac{5}{3}\,[0+2(2.1+1.6)+4(1.2+2.4+0.6)]$

$$= \frac{5}{3}\,(7.4+16.8) = \frac{5}{3}\,(24.2) = \frac{121}{3} \text{ (Simpson's rule).}$$

E16) $S = 2(5+18+25+29+32+20+11+5+2)$

$$= 2(147) = 294.$$

UNIT 16 FURTHER APPLICATIONS OF INTEGRAL CALCULUS

Structure

16.1 INTRODUCTION

In the last unit we have seen how definite integrals can be used to calculate areas. In fact, this application of definite integrals is not surprising. Because, as we have seen earlier, the problem of finding areas was the motivation behind the definition of integrals. In this unit we shall see that the length of an arc of a curve, the volume of a cone and other solids of revolution, the area of a sphere and other surfaces of revolution, can all be expressed as definite integrals. This unit also brings us to the end of this course on calculus.

Objectives

After reading this unit, you should be able to:

- find the length of an arc of a given curve whose equation is expressed in either the Cartesian or parametric or polar forms,
- find the volumes of some solids of revolution,
- find the areas of some surfaces of revolution.

16.2 LENGTH OF A PLANE CURVE

In this section we shall see how definite integrals can be used to find the lengths of plane curves whose equations are given in the Cartesian, polar or parametric form. A curve whose length can be found is called a **rectifiable** curve and the process of finding the length of a curve is called **rectification**. You will see here that to find the length of an arc of a curve, we shall have to integrate an expression which involves not only the given function, but also its derivative. Therefore, to ensure the existence of the integral which determines the arc length, we make an assumption that the function defining the curve is derivable, and its derivative is also continuous on the interval of integration.

Let's first consider a curve whose equation is given in the Cartesian form.

16.2.1 Cartesian Form

Let $y = f(x)$ be defined on the interval $[a, b]$. We assume that f is derivable and its derivative f' is continuous. Let us consider a partition P_n of $[a, b]$, given by

$P_n = \{a = x_0 < x_1 < x_2 < \ldots\ldots < x_n = b\}$

The ordinates $x = a$ and $x = b$ determine the extent of the arc AB of the curve $y = f(x)$ [Fig. 1(a)]. Let M_i, $i = 1, 2, \ldots$, $n-1$, be the points at which the lines $x = x_i$ meet the curve.

Join the successive points A, M_1, M_2, M_3, ..., M_{n-1}, B by straight line segments. Here we have approximated the given curve by a series of line segments.

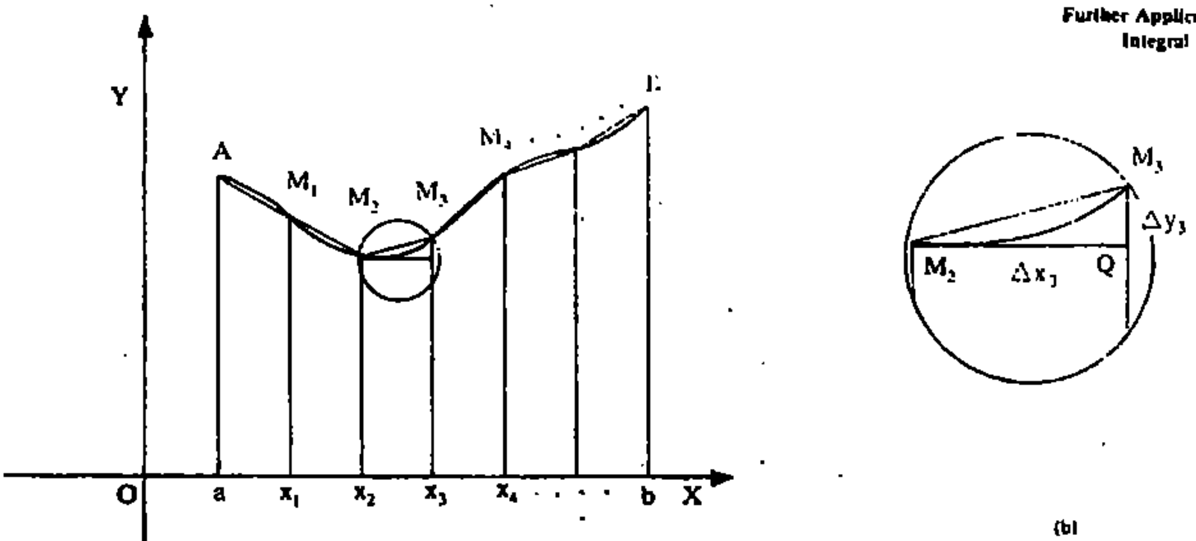


Fig. 1

If we can find the length of each line segment, the total length of this series will give us an approximation to the length of the curve. But how do we find the length of any of these line segments? Take $M_2 M_3$, for example. Fig. 1(b) shows an enlargement of the encircled portion in Fig. 1(a). Looking at it we find that

$M_2M_3 = \sqrt{(\Delta x_3)^2 + (\Delta y_3)^2}$,

where $\Delta x_3 = M_2Q$ is the length $(x_3 - x_2)$, and

$\Delta y_3 = M_3Q = f(x_3) - f(x_2) = y_3 - y_2$.

In this way we can find the lengths of the chords AM_1, M_1M_2, ..., $M_{n-1}B$, and take their sum

$$S_n = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

S_n gives an approximation to the length of the arc AB. When the number of division points is increased indefinitely, and the length of each segment tends to zero, we obtain the length of the arc AB as

$$L_A^B = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}, \quad \text{....(I)}$$

provided this limit exists.

Our assumptions that f is derivable on [a, b], and that f′ is continuous, permit us to apply the mean value theorem [Theorem 3, Unit 7].

Thus, there exists a point $P_i^*(x_i^*, y_i^*)$ between the points M_{i-1} and M_i on the curve, where the tangent to the curve is parallel to the chord $M_{i-1}M_i$. That is,

$$f'(x_i^*) = \frac{\Delta y_i}{\Delta x_i}$$

or $\Delta y_i = f'(x_i^*)\ \Delta x_i$

Hence we can write (I) as

$$L_B^A = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + [f'(x_i^*)\ \Delta x_i]^2}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x_i$$

This is nothing but the definite integral

$$\int_a^b \sqrt{(1 + [f'(x)]^2]}\, dx.$$

Therefore

$$L_A^B = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \ldots(2)$$

Remark 1: It is sometimes convenient to express x as a single valued function of y. In this case we interchange the roles of x and y, and get the length

$$L_A^B = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy, \qquad \ldots(3)$$

where the limits of integration are with respect to y. Note that the length of an arc of a curve is invariant since it does not depend on the choice of coordinates, that is, on the frame of reference. Our assumption that f′ is continuous on [a, b] ensures that the integrals in (2) and (3) exist, and their value L_A^B is the length of the curve y = f(x) between the ordinates x = a and x = b.

The following example illustrates the use of the formulas given by (2) and (3).

Example 1: Suppose we want to find the length of the arc of the curve y = ln x intercepted by the ordinates x = 1 and x = 2.

We have drawn the curve y = ln x in Fig. 2.

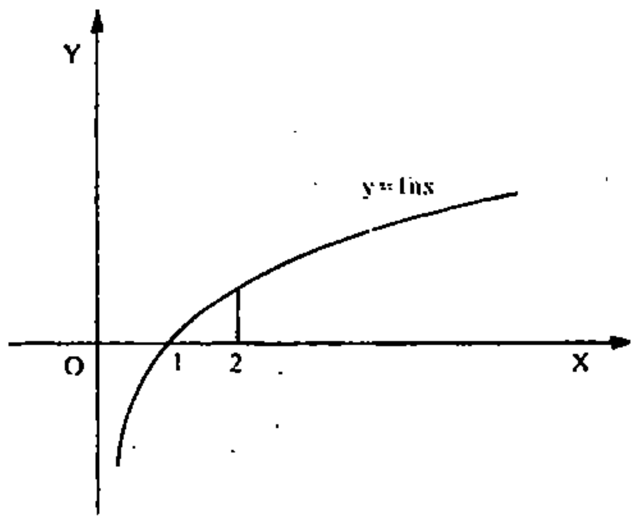


Fig. 2

Using (2), the required length L_1^2 is given by

$$L_1^2 = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\left(1+\frac{1}{x^2}\right)}\, dx, \text{ since } \frac{dy}{dx} = \frac{1}{x}$$

$$= \int_1^2 \frac{\sqrt{1+x^2}}{x}\, dx$$

If we put $1+x^2 = t^2$, we get $\frac{dx}{dt} = \frac{t}{x}$, and therefore,

$$L_1^2 = \int_{\sqrt{2}}^{\sqrt{5}} \left(1+\frac{1}{(t^2-1)}\right) dt$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} dt + \int_{\sqrt{2}}^{\sqrt{5}} \frac{1}{t^2 - 1}\, dt$$

$$= \left[t + \frac{1}{2} \ln \frac{t-1}{t+1} \right]_{\sqrt{2}}^{\sqrt{5}}$$

$$= \sqrt{5} - \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{5}-1}{\sqrt{5}+1} - \frac{1}{2} \ln \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$= \sqrt{5} - \sqrt{2} + \ln \frac{2}{\sqrt{5}+1} - \ln \frac{1}{\sqrt{2}+1}$$

$$= \sqrt{5} - \sqrt{2} + \ln \frac{2(\sqrt{2}+1)}{\sqrt{5}+1}$$

We can also use (3) to solve this example. For this we write the equation $y = \ln x$ as $x = e^y$. The limits $x = 1$ and $x = 2$, then correspond to the limits $y = 0$ and $y = \ln 2$, respectively. Hence, using (3), we obtain

$$L_0^{\ln 2} = \int_0^{\ln 2} \sqrt{1 + e^{2y}}\, dy$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \frac{u^2}{u^2 - 1}\, du, \text{ on putting } 1 + e^{2y} = u^2$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \left(1 + \frac{1}{u^2 - 1}\right) du, = \sqrt{5} - \sqrt{2} + \ln \frac{2(\sqrt{2}+1)}{\sqrt{5}+1},$$

as we have seen earlier. This verifies our observation in Remark 1 that both (2) and (3) give us the same value of arc length.

Now here are some exercises for you to solve.

E E 1) Find the length of the line $x = 3y$ between the points (3, 1) and (6, 2). Verify your answer by using the distance formula.

E 2) Find the length of the curve $y = \ln \sec x$ between the points $x = 0$ and $x = \pi/3$.

E 3) Find the length of the arc of the catenary $y = C \cosh (x/c)$ measured from the vertex $(0, c)$ to any point (x, y) on the catenary.

E 4) Find the length of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a).

E 5) Show that the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is
$a(\ln 2 + 15/16)$.

In the next sub-section we shall consider curves whose equations are expressed in the parametric form.

16.2.2 Parametric Form

Sometimes the equation of a curve cannot be written either in the form $y = f(x)$ or in the form $x = g(y)$. A common example is a circle $x^2 + y^2 = a^2$. In such cases we try to write the equation of the curve in the parametric form. For example, the above circle can be represented by the pair of equations $x = a \cos t$, $y = a \sin t$. Here we shall derive a formula to find the length of a curve given by a pair of parametric equations.

Let $x = \phi(t)$, $y = \psi(t)$, $\alpha \le t \le \beta$ be the equation of a curve in parametric form. As in the previous sub-section, we assume that the functions ϕ and ψ are both derivable and have continuous derivatives ϕ' and ψ' on the interval $[\alpha, \beta]$. We have

$$\frac{dx}{dt} = \phi'(t), \text{ and } \frac{dy}{dt} = \psi'(t).$$

Hence, $\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}$, and

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{\psi'(t)}{\phi'(t)}\right)^2}$$

$$= \frac{\sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}}{\phi'(t)} \quad \text{(we assume that } \phi'(t) \neq 0\text{)}.$$

Now, using (3), we obtain the length

$$L = \int_{x=\phi(\alpha)}^{x=\phi(\beta)} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{t=\alpha}^{t=\beta} \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\; \frac{\phi'(t)}{\phi'(t)}\, dt$$

Thus, $L = \int_\alpha^\beta \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt$ ....(4)

The following example shows that sometimes it is more convenient to express the equation of a given curve in the parametric form in order to find its length.

Example 2: Let us find the whole length of the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$$

By substitution, you can easily check that $x = a\cos^3 t$, $y = b\sin^3 t$ is the parametric form of the given curve. The curve lies between the lines $x = \pm a$ and $y = \pm b$ since $-1 \le \cos t \le 1$, and $-1 \le \sin t \le 1$. The curve is symmetrical about both the axes since its equation remains unchanged if we change the signs of x and y. The value $t = 0$ corresponds to the point (a, 0) and $t = \pi/2$ corresponds to the point (0, b). By applying the curve tracing methods discussed in Unit 9 we can draw this curve (see Fig. 3).

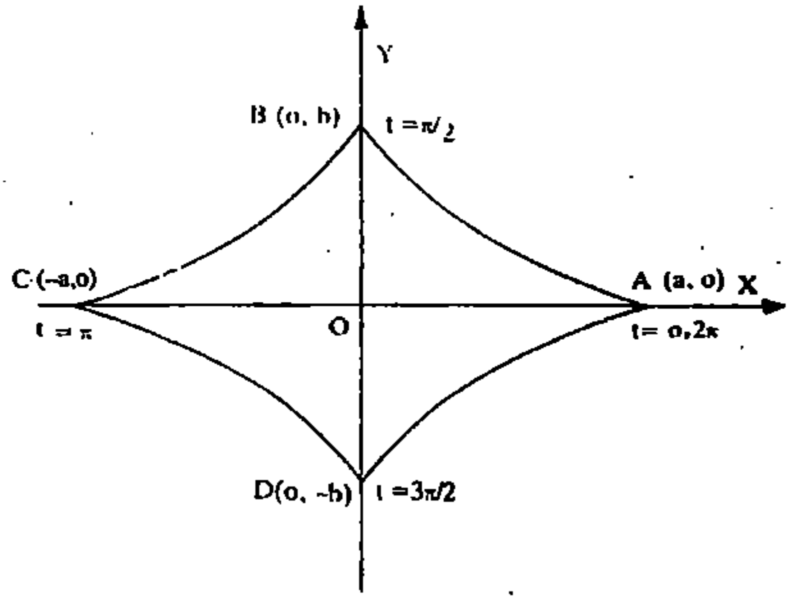


Fig. 3

Since the curve is symmetrical about both axes, the total length of the curve is four times its length in the first quadrant.

Now, $\frac{dx}{dt} = -3a\cos^2 t \sin t$; $\frac{dy}{dt} = 3b\sin^2 t\cos t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t\cos^2 t\,(a^2\cos^2 t + b^2\sin^2 t)$$

Hence, the length of the curve is

$$L = 4\int_0^{\pi/2} 3\sin t\cos t\,\sqrt{a^2\cos^2 t + b^2\sin^2 t}\;dt$$

$$= 12\int_0^{\pi/2} \sin t\cos t\,\sqrt{a^2\cos^2 t + b^2\sin^2 t}\;dt$$

Putting $u^2 = a^2\cos^2 t + b^2\sin^2 t$, we obtain

$$2u = (2b^2 - 2a^2)\sin t\cos t\,\frac{dt}{du},$$

and the limits $t = 0$, $t = \pi/2$ correspond to $u = a$, $u = b$, respectively.

Thus, we have

$$L = 12\int_a^b \frac{u^2\,du}{b^2-a^2} = \frac{12}{b^2-a^2}\left[\frac{u^3}{3}\right]_a^b$$

$$= \frac{12}{b^2-a^2}\cdot\frac{b^3-a^3}{3} = \frac{4(a^2+b^2+ab)}{a+b}$$

Now you can apply equation (4) to solve these exercises.

E 6) Find the length of the cycloid
$x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$

E 7) Show that the length of the arc of the curve
$x = e^t \sin t$, $y = e^t \cos t$ from $t = 0$ to $t = \pi/2$ is $\sqrt{2}(e^{\pi/2} - 1)$.

16.2.3 Polar Form

In this sub-section we shall consider the case of a curve whose equation is given in the polar form.

Let $r = f(\theta)$ determine a curve as θ varies from $\theta = \alpha$ to $\theta = \beta$, i.e., the function f is defined in the interval $[\alpha, \beta]$ (see Fig. 4). As before, we assume that the function f is derivable and its derivative f′ is continuous on $[\alpha, \beta]$. This assumption ensures that the curve represented by $r = f(\theta)$ is rectifiable.

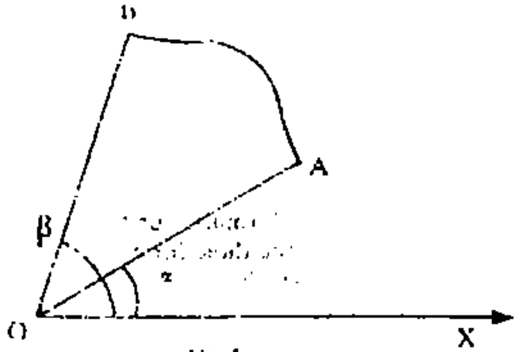


Fig. 4

Transforming the given equations into Cartesian coordinates by taking $x = r\cos\theta$, $y = r\sin\theta$, we obtain $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$.

Now we proceed as in the case of parametric equations, and get

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}}{dx/d\theta}$$

Hence, the length of the arc of the curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{x=f(\alpha)\cos\alpha}^{x=f(\beta)\cos\beta} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta,$$

changing the variable x to θ.

$$= \int_{\alpha}^{\beta} \sqrt{[f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2}\,d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \ldots(5)$$

We shall apply this formula to find the length of the curve in the following example.

Example 3: To find the perimeter of the cardioid $r = a(1+\cos\theta)$ we note that the curve is symmetrical about the initial line (Fig. 5). Therefore its perimeter is double the length of the arc of the curve lying above the x-axis.

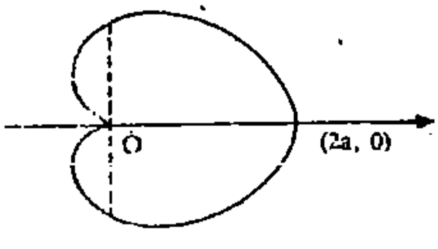


Fig. 5

Now, $\frac{dr}{d\theta} = -a\sin\theta$. Hence, we have

$$L = 2\int_{0}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2a\int_{0}^{\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta}\,d\theta$$

$$= 2a \int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta = 4a \int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$$

$$= 4a \left| 2\sin\frac{\theta}{2} \right|_0^{\pi} = 8a.$$

$\frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2}$

In this section we have derived and applied the formulas for finding the length of a curve when its equation is given in either of the three forms: Cartesian, parametric or polar. Let us summarise our discussion in the following table.

Table 1: Length of an arc of a curve

Equation of the Curve	Length L
$y = f(x)$	$\int_a^b \sqrt{1+f'(x)^2}\, dx$
$x = g(y)$	$\int_c^d \sqrt{1+g'(y)^2}\, dy$
$x = \phi(t),\ y = \psi(t)$	$\int_\alpha^\beta \sqrt{\phi'(t)^2+\psi'(t)^2}\, dt$
$r = f(\theta)$	$\int_\alpha^\beta \sqrt{f(\theta)^2+f'(\theta)^2}\, d\theta$

Using this table you will be able to solve these exercises now.

E E 8) Find the length of the curve $r = a\cos^3(\theta/3)$.

E E 9) Find the length of the circle of radius 2 which is given by the equations $x = 2\cos t + 3,\ y = 2\sin t + 4,\ 0 \le t \le 2\pi$.

E 10) Show that the arc of the upper half of the curve $r = a(1 - \cos\theta)$ is bisected by $\theta = 2\pi/3$.

E 11) Find the length of the curve $r = a(\theta^2 - 1)$ from $\theta = -1$ to $\theta = 1$

16.3 VOLUME OF A SOLID OF REVOLUTION

Until now, in this course, we were concerned with only plane curves and regions. In this section we shall see how our knowledge of integration can be used to find the volumes of certain solids. Look at the plane region in Fig. 6(a). It is bounded by $x = a$, $x = b$, $y = f(x)$ and the x-axis. If we rotate this plane region about the x-axis, we get a solid. See Fig. 6(b).

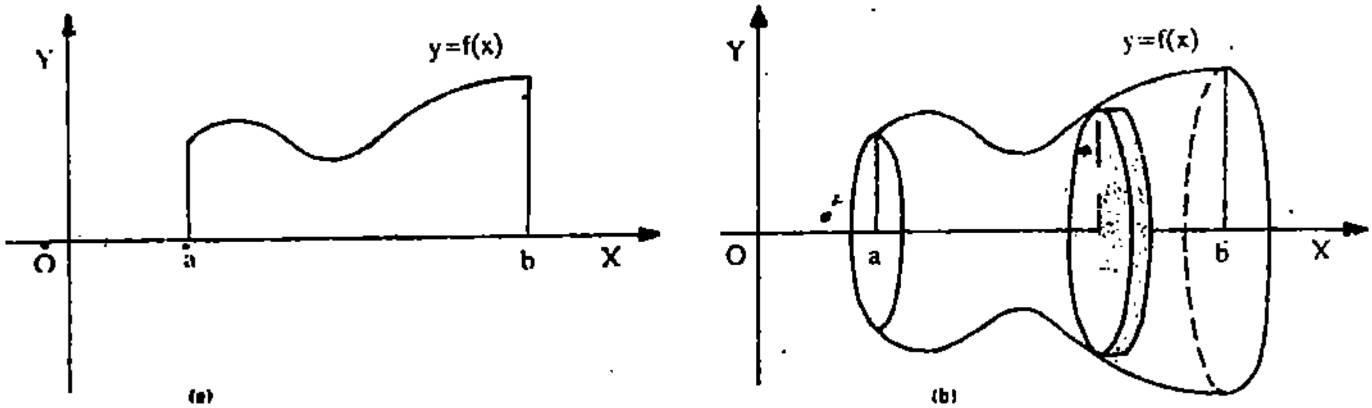


Fig. 6

Such solids are called solids of revolution. Fig. 7(a) and Fig. 7(b) show two more examples of solids of revolution.

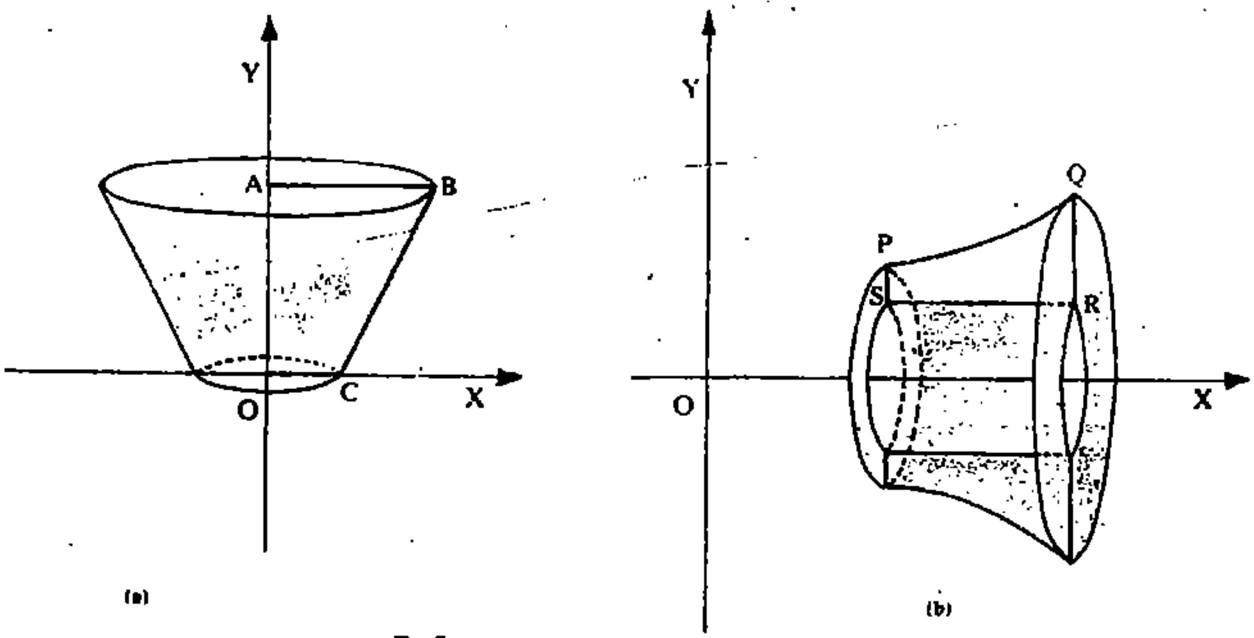
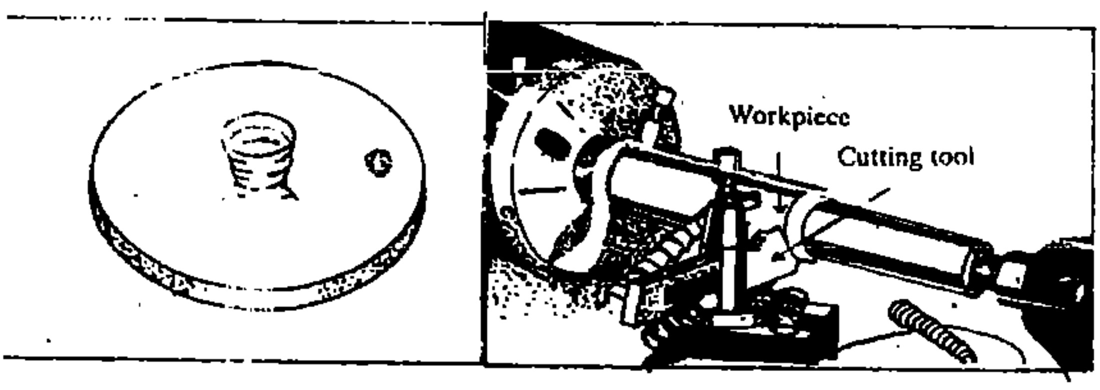


Fig. 7

The solid in Fig. 7(a) is obtained by revolving the region ABCO around the y-axis. The solid of revolution in Fig. 7(b) differs from the others in that its axis of rotation does not form a part of the boundary of the plane region PQRS which is rotated.

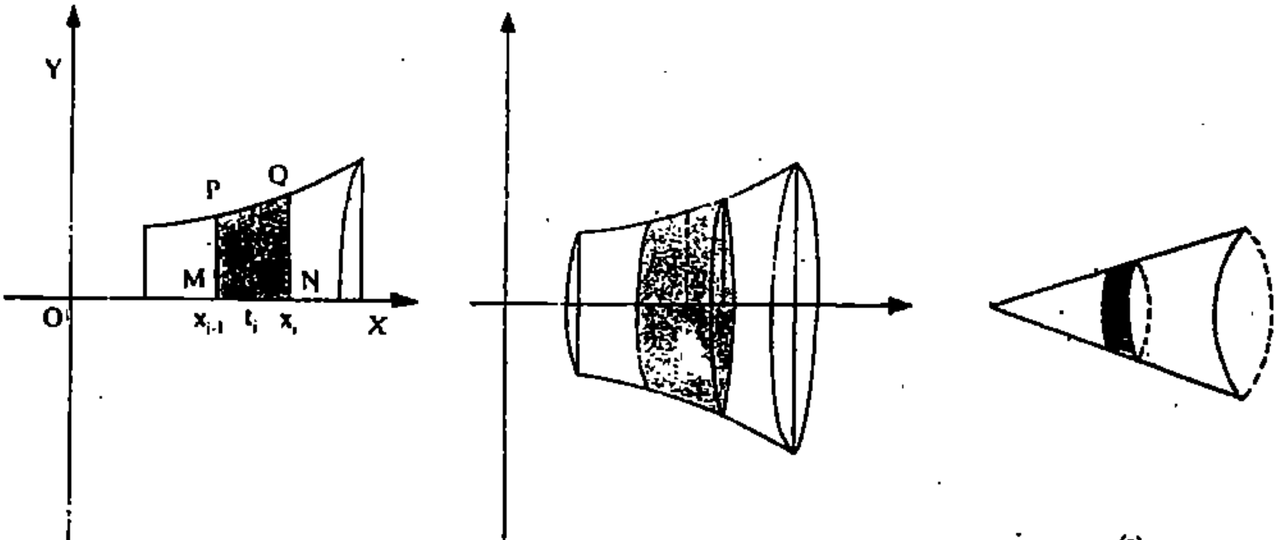
We see many examples of solids of revolution in every day life. The various kinds of pots made by a potter using his wheel are solids of revolution: See Fig. 8(a). Some objects manufactured with the help of lathe machines are also solids of revolution. See Fig. 8(b).



(a) Fig. 8 (b)

Now, let us try to find the volume of a solid of revolution. The method which we are going to use is called the method of slicing. The reason for this will be clear in a few moments.

Let $T_n = \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1}, x_n = b\}$ be a partition of the interval [a, b] into n sub-intervals.



(a) (b) Fig. 9 (c)

Let Δx_i denote the length of the ith sub-interval $[x_{i-1}, x_i]$. Further, let P and Q be the points on the curve, $y = f(x)$ corresponding to the ordinates $x = x_{i-1}$ and $x = x_i$, respectively. Then, as the curve revolves about the x-axis, the shaded strip PQNM (Fig. 9(a)) generates a disc of thickness Δx_i. In general, the ordinates PM and QN may not be of equal length. Hence, the disc is actually the frustum of a cone with its volume Δv_i, lying between $\pi\, PM^2 MN$ and $\pi\, QN^2 MN$, that is, between

$\pi\,[f(x_{i-1})]^2 \Delta x_i$ and $\pi\,[f(x_i)]^2 \Delta x_i$ (Fig. 9(b) and (c))

πPM^2. MN is the volume of the disc with radius PM and thickness MN.
$\pi QN^2 MN$ is the volume of the disc with radius QN and thickness MN.

If f is continuous on [a, b], $f(a) = c$ and $f(b) = d$, and z lies between c and d, then $\exists x_0 \in]a, b[$ s.t. $f(x_0) = z$.

If we assume that f is a continuous function on [a, b], we can apply the intermediate value theorem (Theorem 7, Unit 2, also see margin remark), and express this volume as

$\Delta v_i = \pi\{f(t_i)\}^2\ \Delta x_i$, where t_i is a suitable point in the interval $[x_{i-1}, x_i]$. Now summing up over all the discs, we obtain

$$V_n = \sum_{i=1}^{n} \Delta v_i = \sum_{1}^{n} \pi\,[f(t_i)]^2\ \Delta x_i, \quad x_{i-1} \le t_i \le x_i, \text{ as an approximation}$$

to the volume of the solid of revolution. As we have observed earlier while defining a definite integral, the approximation gets better as the partition P_n gets finer and finer and Δx_i tends to zero. Thus, we get the volume of the solid of revolution as

$$V = \lim_{n \to \infty} V_n = \lim_{n \to \infty} \sum_{i=1}^{n} \pi\,[f(t_i)]^2\, \Delta x_i$$

$$= \pi \int_a^b [f(x)]^2\, dx = \pi \int_a^b y^2\, dx \qquad \ldots(6)$$

We shall use this formula to find the volume of the solid described in the following example.

Example 4: Let us find the volume of the solid of revolution formed when the arc of the parabola $y^2 = 4ax$ between the ordinates $x = 0$, and $x = a$ is revolved about its axis. The solid of revolution is the parabolic cap in Fig. 10.

The volume V of the cap is given by

$$V = \int_0^a \pi y^2\, dx = \pi \int_0^a 4ax\, dx = 4\pi a \left[\frac{x^2}{2}\right]_0^a = 2\pi a^3 \cdot$$

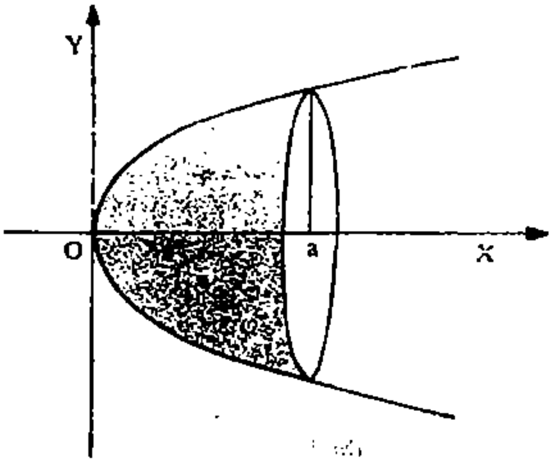


Fig. 10

Our next example illustrates a slight modification of Formula (6) to find the volume of a solid obtained by revolving a plane region about the y-axis.

Example 5: Suppose the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a > b)$ is revolved about the minor axis, AB (see Fig. 11). Let us find the volume of the solid generated.

In this case the axis of rotation is the y-axis. The area revolved about the y-axis is shown by the shaded region in Fig. 11. You will agree that we need to consider only the area to the right of the y-axis.

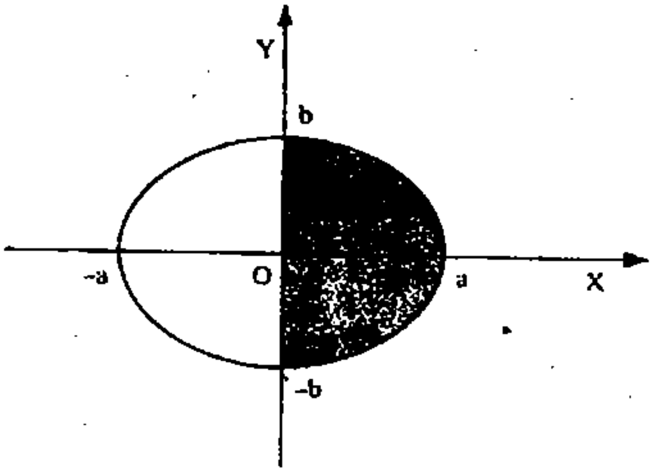


Fig. 11

To find the volume of this solid we interchange x and y in (6) and get

$$V = \int_{-b}^{b} \pi x^2\, dy = \int_{-b}^{b} \pi a^2 \left(1 - \frac{y^2}{b^2}\right) dy, \text{ since } x^2 = a^2\left(1 - \frac{y^2}{b^2}\right).$$

$$= 2\pi a^2 \int_0^b \left(1 - \frac{y^2}{b^2}\right) dy \text{ since } 1 - \frac{y^2}{b^2} \text{ is an even function of y.}$$

$$= 2\pi a^2 \left[y - \frac{y^3}{3b^2}\right]_0^b$$

$$= \frac{4}{3}\pi a^2 b.$$

We can also modify Formula (6) to apply to curves whose equations are given in the parametric or polar forms. Let us tackle these one by one.

Parametric Form

If a curve is given by $x=\phi(t)$, $y=\psi(t)$, $\alpha \le t \le \beta$, then the volume of the solid of revolution about the x-axis can be found by substituting x and y in Formula (6) by $\phi(t)$ and $\psi(t)$ respectively. Thus,

$$V = \pi\int_{\alpha}^{\beta} [\psi(t)]^2 \frac{dx}{dt}\, dt$$

$$\text{or } V = \pi \int_{\alpha}^{\beta} [\psi(t)]^2 \phi'(t)\, dt.$$

We'll now derive the formula for curves given by polar equations.

Polar Form

Suppose a curve is given by $r=f(\theta)$, $\theta_1 \le \theta \le \theta_2$. The volume of the solid generated by rotating the area bounded by $x=a$, $x=b$, the x-axis and $r=f(\theta)$ about the x-axis is

$$V = \pi \int_{\theta_1}^{\theta_2} (r \sin\theta)^2 \frac{d}{d\theta} (r\cos\theta)\, d\theta$$

$$\text{Thus, } V = \pi \int_{\theta_1}^{\theta_2} [f(\theta)\sin\theta]^2 [f'(\theta)\cos\theta - f(\theta)\sin\theta]\, d\theta$$

Let's use this formula to find the volume of the solid generated by a cardioid about its initial line.

Example 6: The cardioid shown in Fig. 12 is given by $r=a(1+\cos\theta)$.

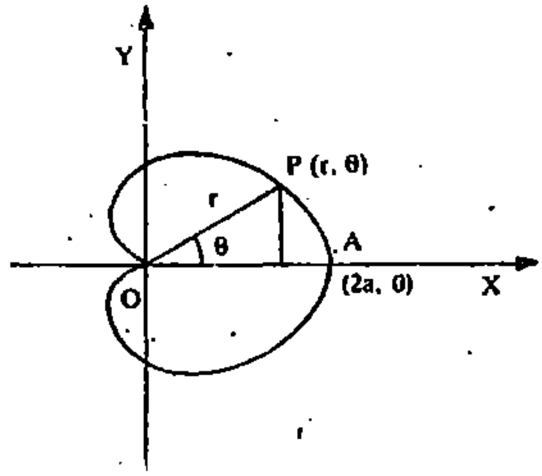


Fig. 12

The points A and O correspond to $\theta=0$ and $\theta=\pi$, respectively. Here, again, we need to consider only the part of the cardioid above the initial line. Thus,

$$V = \int_{\pi}^{0} \pi (r\sin\theta)^2 \frac{d}{d\theta} (r\cos\theta)\, d\theta$$

$$= \pi a^3 \int_{0}^{\pi} (1+\cos\theta)^2 \sin^3\theta\, (1+2\cos\theta)\, d\theta, \text{ since } r=a(1+\cos\theta)$$

$$= \pi a^3 \int_0^{\pi} 8 \sin^3 \frac{\theta}{2} \cos^3 \frac{\theta}{2}\, 4 \cos^4 \frac{\theta}{2} \left(4 \cos^2 \frac{\theta}{2} - 1\right) d\theta$$

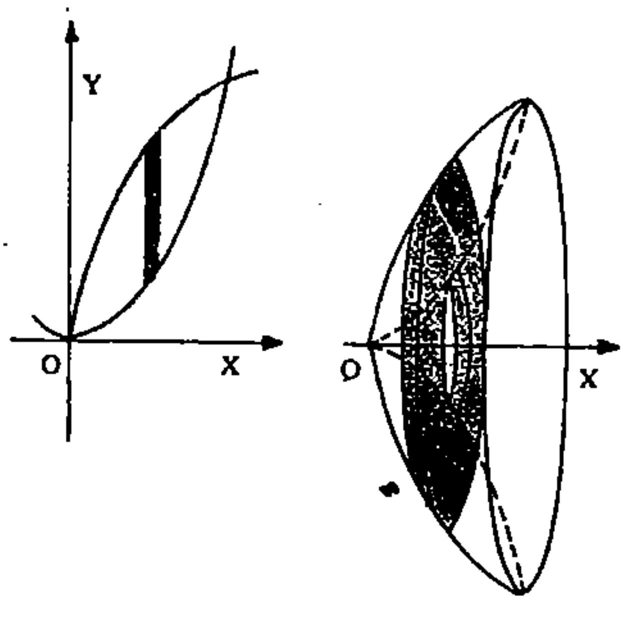
$$= 128\pi a^3 \int_0^{\pi} \sin^3 \frac{\theta}{2} \cos^9 \frac{\theta}{2}\, d\theta - 32\pi a^3 \int_0^{\pi} \sin^3 \frac{\theta}{2} \cos^7 \frac{\theta}{2}\, d\theta$$

$$= 256\pi a^3 \int_0^{\pi/2} \sin^3\phi \cos^9\phi\, d\phi - 64\pi a^3 \int_0^{\pi/2} \sin^3\phi \cos^7\phi\, d\phi, \text{ where } \phi = \theta/2$$

$$= \frac{64\pi a^3}{15} - \frac{8\pi a^3}{5} \text{ on applying a reduction formula from Unit 12.}$$

In all the examples that we have seen till now, the axis of rotation formed a boundary of the region which was rotated. Now we take an example in which the axis touches the region at only one point.

Example 7: Let us find the volume of the solid generated by revolving the region bounded by the parabolas $y = x^2$ and $y^2 = 8x$ about the x-axis. We have shown the area rotated and the solid in Fig. 13 (a) and (b), respectively.



(a) Fig. 13 (b)

Here, the required volume will be the difference between the volume of the solid generated by the parabola $y^2 = 8x$ and that of the solid generated by the parabola $y = x^2$.

Thus, $V = \pi \left[\int_0^2 8x\, dx - \int_0^2 x^4\, dx \right] = \pi \left[4x^2 - \frac{x^5}{5} \right]_0^2 = \frac{48\pi}{5}$

Note the limits of integration.

Here, we list some exercises which you can solve by applying the formulas derived in this section.

E 12) Find the volume of the right circular cone of height h and radius of the circular base r.

(*Hint*: This cone will be generated by rotating the triangle bounded by the x-axis and the line $y = (r/h)\, x$).

E 13) Show that the volume of the solid generated by revolving the curve $x^{2/3}+y^{2/3}=a^{2/3}$ about the x-axis is $32\pi a^3/105$.

E 14) The arc of the cycloid $x=a\,(t-\sin t)$, $y=a\,(1-\cos t)$ in $[0, 2\pi]$ is rotated about the y-axis. Find the volume generated.
(*Caution*: The rotation is about the y-axis.)

E 15) Find the volume of the solid obtained by revolving the limacon $r = a + b\cos\theta$ about the initial line.

E 16) The semicircular region bounded by $y-2=\sqrt{9-x^2}$ and the line $y=2$ is rotated about the x-axis. Find the volume of the solid generated.

16.4 AREA OF SURFACE OF REVOLUTION

Instead of rotating a plane region, if we rotate a curve about an x-axis, we shall get a surface of revolution. In this section we shall find a formula for the area of such a surface. Let us start with the case when the equation of the curve is given in the Cartesian form.

Cartesian Form

Suppose that the curve $y=f(x)$ [Fig. 14] is rotated about the x-axis. To find the area of the generated surface, we consider a partition P_n of the interval [a, b], namely,

$P_n = \{a=x_0 < x_1 < x_2 < \ldots\ldots < x_{n-1} < x_n=b\}$

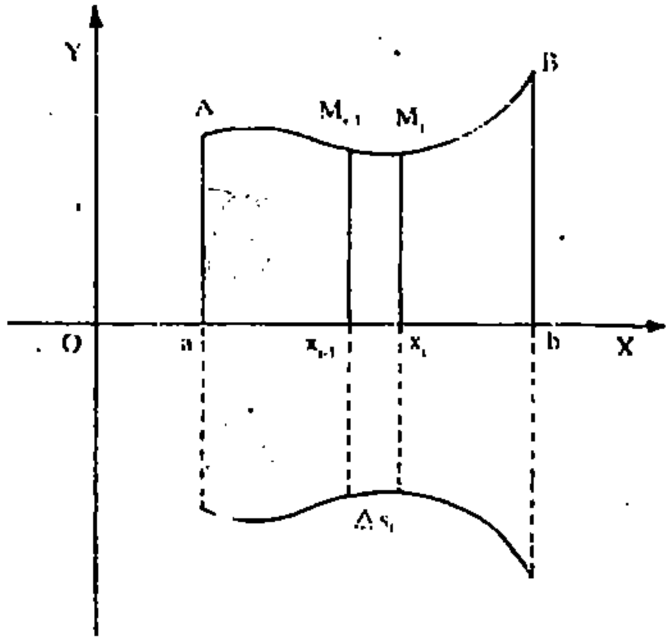


Fig. 14

Let the lines $x = x_i$ intersect the curve in points M_i, $i = 1, 2, \ldots, n$. If we revolve the chord $M_{i-1}M_i$ about the x-axis, we shall get the surface of the frustum of a cone of thickness $\Delta x_i = x_i - x_{i-1}$. Let Δs_i be the area of the surface of this frustum. Then the total surface area of all the frusta is

$$S_n = \sum_{i=1}^{n} \Delta s_i$$

Fig. 15

This S_n is an approximation to the area of the surface of revolution. The area of the surface of revolution generated by the curve $y = f(x)$, is the limit of S_n (if it exists), as $n \to \infty$ and each $\Delta x_i \to 0$.

To find the area A of the curved surface of a typical frustum, we use the formula
$A = \pi(r_1 + r_2)\, l$,
where l is the slant height of the frustum and r_1 and r_2 are the radii of its bases (Fig. 15).

In the frustum under consideration the radii of the bases are the ordinates $f(x_{i-1})$ and $f(x_i)$, while the slant height $M_{i-1}M_i$ is given by $\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$, where $\Delta y_i = f(x_i) - f(x_{i-1})$. We assume that f is derivable on [a, b] and f' is continuous. Then by the mean value theorem (Theorem 3, Unit 7), we obtain

$\Delta y_i = f'(t_i)\, \Delta x_i$, for some $t_i \in [x_{i-1}, x_i]$.

Therefore,

$$\Delta s_i = \pi\, [f(x_{i-1}) + f(x_i)]\, \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= 2\pi \frac{y_{i-1} + y_i}{2} \sqrt{1 + [f'(t_i)]^2}\, \Delta x_i.$$

and $$S_n = 2\pi \sum_{i=1}^{n} \frac{y_{i-1} + y_i}{2} \sqrt{1 + [f'(t_i)]^2}\, \Delta x_i.$$

Proceeding to the limit as $n \to \infty$, and each $\Delta x_i \to 0$, we have

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

$$= 2\pi \int_a^b y \sqrt{1 + (dy/dx)^2}\, dx \qquad \ldots(7)$$

We shall now illustrate the use of this formula.

Example 8: Let us find the area of the surface of revolution obtained by revolving the parabola $y^2 = 4ax$ from $x = a$ to $x = 3a$, about the x-axis.

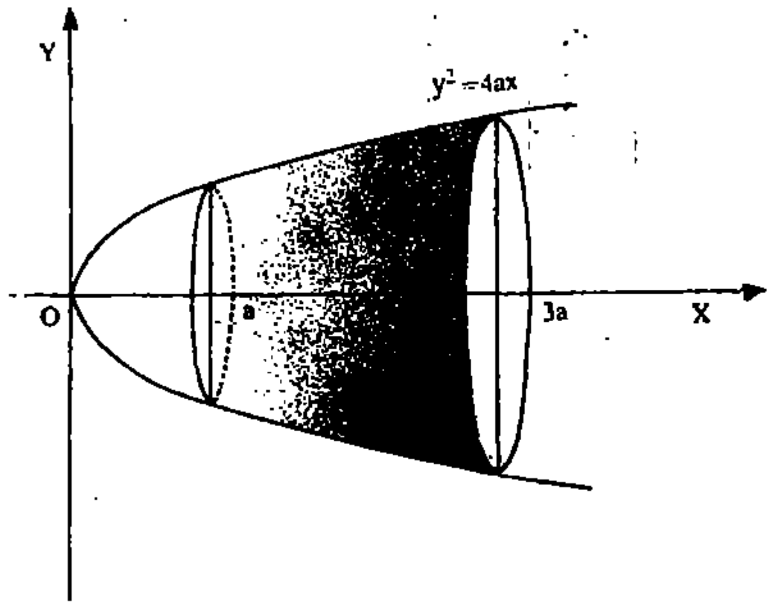


Fig. 16

The area of the surface of revolution

$$S = 2\pi \int_a^{3a} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

where $y^2 = 4ax$, $\frac{dy}{dx} = \frac{2a}{y}$. Hence

$$S = 2\pi \int_a^{3a} y \sqrt{1 + 4a^2/y^2}\, dx$$

$$= 2\pi \int_a^{3a} \sqrt{y^2 + 4a^2}\, dx = 2\pi \int_a^{3a} \sqrt{4ax + 4a^2}\, dx$$

$$= 4\pi \sqrt{a} \int_a^{3a} \sqrt{x + a}\, dx = 4\pi \sqrt{a} \cdot \frac{2}{3} \left[(x+a)^{3/2}\right]_a^{3a}$$

$$= \frac{8\pi a^2}{3} \left[4^{3/2} - 2^{3/2}\right]$$

Instead of revolving the given curve about the x-axis, if we revolve it about the y-axis, we get another surface of revolution. The area of the surface of revolution generated by the curve $x = g(y)$, $c \le y \le d$, as it revolves about the y-axis is given by.

$$S = 2\pi \int_c^d x \sqrt{1 + (dx/dy)^2}\, dy$$

Now let us look at curves represented by parametric equations.

Parametric Form

Suppose a curve is given by the parametric equations $x = \phi(t)$, $y = \psi(t)$, $t \in [\alpha, \beta]$. Then we know that

$$\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}$$

Substituting this in formula (10), we get the area of the surface of revolution generated by the curve as it revolves about the x-axis, to be

$$S = 2\pi \int_{\alpha}^{\beta} \psi(t) \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt.$$

Now we shall state the formula for the surface generated by a curve represented by a polar equation.

Polar Form

If $r = h(\theta)$ is the polar equation of the curve, then the area of the surface of revolution generated by the arc of the curve for $\theta_1 \le \theta \le \theta_2$, as it revolves about the initial line, is

$$S = 2\pi \int_{\theta_1}^{\theta_2} r \sin\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$$

Study the following examples carefully before trying the exercises given at the end of this section.

Example 9: Suppose the astroid $x = a\sin^3 t$, $y = a\cos^3 t$, is revolved about the x-axis. Let us find the area of the surface of revolution. You will agree that we need to consider only the portion of the curve above the x-axis.

For this portion $y > 0$, and thus t varies from $-\pi/2$ to $\pi/2$.

$$\frac{dx}{dt} = 3a\sin^2 t\cos t,\ \frac{dy}{dt} = -3a\cos^2 t\sin t$$

Therefore, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\sin^2 t\cos^2 t$

We therefore get,

$$S = 2\pi \int_{-\pi/2}^{\pi/2} a\cos^3 t\sqrt{9a^2\sin^2 t\cos^2 t}\, dt$$

$$= 2\pi \int_{-\pi/2}^{\pi/2} a\cos^3 t\,|3a\sin t\cos t|\, dt$$

$$= 6\pi a^2 \int_{-\pi/2}^{\pi/2} \cos^4 t\,|\sin t|\, dt$$

$$= 12\pi a^2 \int_{0}^{\pi/2} \cos^4 t\sin t\, dt = -12\pi a^2\left[\frac{\cos^5 t}{5}\right]_0^{\pi/2}$$

$$= \frac{12}{5}\pi a^2$$

Example 10: Suppose we want to find the area of the surface generated by revolving the cardioid $r = a(1+\cos\theta)$ about its initial line.

Notice that the cardioid is symmetrical about the initial line, and extends above this line from $\theta = 0$ to $\theta = \pi$. The surface generated by revolving the whole curve about the initial line is the same as that generated by the upper half of the curve. Hence

$$S = 2\pi \int_{0}^{\pi} r\sin\theta\sqrt{r^2 + (dr/d\theta)^2}\, d\theta$$

$$= 2\pi \int_0^\pi a(1+\cos\theta)\sin\theta\sqrt{r^2+(dr/d\theta)^2}\,d\theta$$

Since $r = a(1+\cos\theta)$, and $\frac{dr}{d\theta} = -a\sin\theta$, we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta = 4a^2\cos^2\frac{\theta}{2}$$

Therefore,

$$S = 2\pi \int_0^\pi a(1+\cos\theta)\sin\theta\, 2a\cos\frac{\theta}{2}\,d\theta$$

$$= 4\pi a^2 \int_0^\pi 4\sin\frac{\theta}{2}\cos^4\frac{\theta}{2}\,d\theta$$

$$= 32\pi a^2 \int_0^{\pi/2} \sin\phi\cos^4\phi\,d\phi, \text{ where } \phi = \theta/2$$

$$= 32\pi a^2 \left[\frac{-\cos^5\phi}{5}\right]_0^{\pi/2} = \frac{32\pi a^2}{5}$$

E E 17) Find the area of the surface generated by revolving the circle $r = a$ about the x-axis, and thus verify that the surface area of a sphere of radius a is $4\pi a^2$.

E E 18) The arc of the curve $y = \sin x$, from $x = 0$ to $x = \pi$ is revolved about the x-axis. Find the area of the surface of the solid of revolution generated.

E 19) The ellipse $x^2/a^2 + y^2/b^2 = 1$ revolves about the x axis. Find the area of the surface of the solid of revolution thus obtained.

E 20) Prove that the surface of the solid generated by the revolution about the x-axis of the loop of the curve $x = t^2$,

$$y = t - \frac{t^3}{3} \text{ is } 3\pi.$$

E 21) Find the surface area of the solid generated by revolving the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, about the line $y = 0$.

Now let us quickly recall what we have covered in this unit.

16.5 SUMMARY

In this unit we have seen how to find

1) the lengths of curves
2) volumes of solids of revolution and
3) the areas of surfaces of revolution.

In each case we have derived formulas when the equation of the curve is given in either the Cartesian or parametric or polar form. We give the results here in the form of the following tables.

Length of an arc of a curve

Equation	Length
$y = f(x)$	$\int_a^b \sqrt{1+[f'(x)]^2}\, dx$
$x = g(y)$	$\int_c^d \sqrt{1+[g'(y)]^2}\, dy$
$x = \phi(t)$ $y = \psi(t)$	$\int_\alpha^\beta \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt$
$r = f(\theta)$	$\int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$

Volume of the solid of revolution

Equation	Volume
$y = f(x)$ about x-axis	$\pi \int_a^b y^2\, dx$
$x = g(y)$ about y-axis	$\pi \int_c^d x^2\, dy$
$x = \phi(t)$, $y = \psi(t)$ about x-axis	$\pi \int_\alpha^\beta [\psi(t)]^2\, \phi'(t)\, dt$
$r = h(\theta)$ about the initial line	$\pi \int_{\theta_1}^{\theta_2} [h(\theta) \sin\theta]^2\, [h'(\theta)\cos\theta - h(\theta)\sin\theta]\, d\theta$

Area of the surface of revolution

Equation	Area
$y = f(x)$ about x-axis	$2\pi \int_a^b f(x) \sqrt{1+[f'(x)]^2}\, dx$
$x = g(y)$ about y-axis	$2\pi \int_c^d g(y) \sqrt{1+[g'(y)]^2}\, dy$
$x = \phi(t),\ y = \psi(t)$ about x-axis	$2\pi \int_\alpha^\beta \psi(t) \sqrt{[\phi'(t)]^2+[\psi'(t)]^2}\, dt$
$r = h(\theta)$ about the initial line	$2\pi \int_{\theta_1}^{\theta_2} r\sin\theta \sqrt{r^2+(dr/d\theta)^2}\, d\theta$

16.6 SOLUTIONS AND ANSWERS

E1) $L = \int_c^d \sqrt{1 + (dx/dy)^2}\, dy$

$= \int_1^2 \sqrt{1 + (3)^2}\, dy$

$= \sqrt{10} \int_1^2 dy = \sqrt{10}.$

By distance formula,

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3 - 6)^2 + (1 - 2)^2}$

$= \sqrt{(-3)^2 + (-1)^2}$

$= \sqrt{10}.$

E2) $L = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx \qquad \left(\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x\right)$

$= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$

$= \int_0^{\pi/3} \sec x\, dx = \ln |\sec x + \tan x| \Big|_0^{\pi/3}$

$= \ln \left| \frac{\sec\pi/3 + \tan \pi/3}{\sec 0 + \tan 0} \right|$

$= \ln (2 + \sqrt{3})$

E3) $L = \int_0^x \sqrt{1 + \sinh^2(x/c)}\, dx$

$= \int_0^x \cosh(x/c)\, dx$

$= c \sinh(x/c)\Big]_0^x = c \sinh(x/c)$

E4) $y = \sqrt{\frac{x^3}{a}} \quad \therefore dy/dx = (3/2)\sqrt{\frac{x}{a}}$

$L = \int_0^a \sqrt{1 + \frac{9x}{4a}}\, dx$

$= \frac{1}{2\sqrt{a}} \int \sqrt{4a + 9x}\; dx$

$= \frac{1}{27\sqrt{a}} (4a + 9x)^{3/2}\Big]_0^a$

$= \frac{1}{27\sqrt{a}} [(13a)^{3/2} - (4a)^{3/2}] = \frac{a}{27}(13^{3/2} - 8)$

E5) $3y = 8x \Rightarrow y = \frac{8x}{3}$. Substituting this in $y^2 = 4ax$ we get

$\frac{64x^2}{9} = 4ax$

i.e. $64x^2 - 36ax = 0$

$\Rightarrow x = 0$ or $x = \frac{9a}{16}$

$\Rightarrow y = 0$ or $y = \frac{3a}{2}$

Hence (0, 0) and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$ are the points of intersectio.

Now $4ax = y^2 \Rightarrow \frac{dx}{dy} = \frac{y}{2a}$

$L = \int_0^{3a/2} \sqrt{1 + \frac{y^2}{4a^2}}\, dy$

$= \frac{1}{2a} \int_0^{3a/2} \sqrt{4a^2 + y^2}\, dy$

$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{4a^2 + y^2} + 2a^2 \ln | y + \sqrt{4a^2 + y^2} |\right]_0^{3a/2}$

$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2 \ln 2\right]$

$= \left(\frac{15}{16} + \ln 2\right) a$

E6) $\frac{dx}{d\theta} = a(1 - \cos\theta)$, $\frac{dy}{d\theta} = a \sin\theta$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2[1 + \cos^2\theta - 2\cos\theta + \sin^2\theta]$$

$$= 2a^2(1 - \cos\theta)$$

$$= 4a^2 \sin^2(\theta/2)$$

$$\therefore L = 2a\int_0^{2\pi} \sin(\theta/2)\, d\theta$$

$$= 4a\int_0^{\pi} \sin\phi\, d\phi$$

$$= 8a\int_0^{\pi/2} \sin\phi\, d\phi = 8a$$

E7) $\frac{dx}{dt} = e^t(\cos t + \sin t),\ \frac{dy}{dt} = e^t(\cos t - \sin t)$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{2t}$$

$$\therefore L = \sqrt{2}\int_0^{\pi/2} e^t\, dt = \sqrt{2}\, e^t \Big|_0^{\pi/2}$$

$$= \sqrt{2}\,(e^{\pi/2} - 1)$$

E8) $r = a\cos^3\frac{\theta}{3} \Rightarrow \frac{dr}{d\theta} = -a\cos^2\frac{\theta}{3}\sin\frac{\theta}{3}$

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos^6\frac{\theta}{3} + a^2\cos^4\frac{\theta}{3}\sin^2\frac{\theta}{3}$$

$$= a^2\cos^4\frac{\theta}{3}$$

$$\therefore L = 2a\int_0^{3\pi/2} \cos^2\frac{\theta}{3}\, d\theta = 6a\int_0^{\pi/2} \cos^2\phi\, d\phi$$

$$= \frac{3a\pi}{2}$$

E9) $\frac{dx}{dt} = -2\sin t\ \frac{dy}{dt} = 2\cos t$

$$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\sqrt{\sin^2 t + \cos^2 t} = 2$$

$$\therefore L = 2\int_0^{2\pi} dt = 4\pi$$

Note that $L = 2\pi r$ since, here, $r = 2$.

E10) $r = a(1 - \cos\theta),\ \frac{dr}{d\theta} = a\sin\theta$

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2a\sin\frac{\theta}{2}$$

The length of the curve in the upper half $= \int_0^{\pi} 2a\sin(\theta/2)\, d\theta$

$= 4a.$

The length from $\theta = 0$ to $\theta = 2\pi/3$

$$= \int_0^{2\pi/3} 2a \sin\frac{\theta}{2}\, d\theta = 2a$$

The arc of the curve in the upper half is bisected by $\theta = 2\pi/3$.

E11) $r = a(\theta^2 - 1)$, $\dfrac{dr}{d\theta} = 2a\theta$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2[\theta^4 - 2\theta^2 + 1 + 4\theta^2]$$

$$= a^2(\theta^2 + 1)^2.$$

$$\therefore L = a\int_{-1}^{1} (\theta^2 + 1)^2\, d\theta$$

$$= a\left(\frac{\theta^3}{3} + \theta\right)\Bigg]_{-1}^{1}$$

$$= a\left(\frac{1}{3} + 1 + \frac{1}{3} + 1\right) = \frac{8a}{3}$$

E12) $V = \pi \displaystyle\int_0^h \frac{r^2}{h^2}\, x^2\, dx$

$$= \frac{\pi r^2}{h^2}\,\frac{x^3}{3}\Bigg|_0^h = \frac{1}{3}\pi r^2 h.$$

E13) $V = 2\pi \displaystyle\int_0^a (a^{2/3} - x^{2/3})^3\, dx$

$$= 2\pi \int_0^a (a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2)\, dx$$

$$= 2\pi\left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - x^3/3\right]_0^a$$

$$= \frac{32\pi a^3}{105}$$

Note that the total volume generated is equal to twice the volume generated by the arc of the curve between $x = 0$ and $x = a$.

[illegible] $(t - \sin t)^2$, $\dfrac{dy}{dt} = a \sin t$

[illegible] $\int$ [illegible] $a \sin t\, dt$

[illegible] $\int$ [illegible] $\sin t + \sin^3 t)\, dt$

[illegible] 4π [illegible] $2\pi^2)$ [illegible].

E15) $r = a + b\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{d(r\cos\theta)}{d\theta}$

$$= (a + b\cos\theta)(-\sin\theta) - b\sin\theta\cos\theta$$

$$= -a\sin\theta - 2b\sin\theta\cos\theta.$$

$$\therefore V = \pi\int_0^{\pi} y^2 \frac{dx}{d\theta}\, d\theta$$

$$= -\pi\int_0^{\pi}(a + b\cos\theta)^2 \sin^2\theta\,(a\sin\theta + 2b\sin\theta\cos\theta)\, d\theta$$

$$= -\pi\int_0^{\pi}(a^3\sin^3\theta + 4a^2b\sin^3\theta\cos\theta + 5ab^2\sin^3\theta\cos^2\theta + 2b^3\sin^3\theta\cos^3\theta)\, d\theta$$

$$= -\pi\left[2a^3\int_0^{\pi/2}\sin^3\theta\, d\theta + 10ab^2\int_0^{\pi/2}\sin^3\theta\cos^2\theta\, d\theta\right]$$

(The other two integrals are equal to zero since $\cos(\pi - \theta) = -\cos\theta$.

$$= -\pi\left[\frac{4a^3}{3} + 10ab^2 \cdot \frac{2}{5} \cdot \int_0^{\pi/2}\sin\theta\cos^2\theta\, d\theta\right]$$

(using a reduction formula)

$$= -\pi\left[\frac{4a^3}{3} - \frac{4ab^2}{3}\right] = \frac{4\pi a}{3}(b^2 - a^2)$$

E16) $V = 2\pi\left[\int_0^3 (2 + \sqrt{9 - x^2})^2\, dx - \int_0^3 4\, dx\right]$

$$= 2\pi\left[\int_0^3 (4 + 9 - x^2 + 4\sqrt{9 - x^2})\, dx - 12\right]$$

$$= 2\pi\left[13x - \frac{x^3}{3} + 2x\sqrt{9 - x^2} + 18\sin^{-1}\frac{x}{3}\right]_0^3 - 24\pi$$

$$= 36\pi + 18\pi^2 = 18\pi(2 + \pi)$$

E17) $r = a \Rightarrow \frac{dr}{d\theta} = 0$

$$S = 2\pi\int_0^{\pi} a\sin\theta\,\sqrt{a^2 + 0}\, d\theta$$

$$= 2\pi a^2\int_0^{\pi}\sin\theta\, d\theta$$

$$= 4\pi a^2\int_0^{\pi/2}\sin\theta\, d\theta \quad \text{since } \sin(\pi - \theta) = \sin\theta$$

$$= 4\pi a^2 \cos\theta\Big|_0^{\pi/2}$$

$$= 4\pi a^2.$$

E18) $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos^2 x$$

$$\therefore S = 2\pi \int_0^{\pi} \sin x \sqrt{1 + \cos^2 x}\, dx$$

$$= 4\pi \int_0^{\pi/2} \sin x \sqrt{1 + \cos^2 x}\, dx$$

$$= 4\pi \int_0^{1} \sqrt{1 + t^2}\, dt, \text{ if } t = \cos x$$

$$= 4\pi \left[\frac{t}{2}\sqrt{1 + t^2} + \frac{1}{2}\ln\left| t + \sqrt{1 + t^2}\right|\right]_0^1$$

$$= 2\sqrt{2}\pi + 2\pi \ln(1 + \sqrt{2}).$$

E19) $y = \frac{b}{a}\sqrt{a^2 - x^2}$.

$$\therefore \frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2 - x^2}}$$

$$S = 4\pi \int_0^{a} \frac{b}{a}\sqrt{a^2 - x^2}\sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}\, dx$$

$$= \frac{4\pi b}{a^2}\int_0^{a}\sqrt{a^4 - (a^2 - b^2)x^2}\, dx$$

$$= \frac{4\pi b\sqrt{a^2 - b^2}}{a^2}\int_0^{a}\sqrt{\frac{a^4}{a^2 - b^2} - x^2}\, dx$$

$$= \frac{4\pi b\sqrt{a^2 - b^2}}{a^2}\left[\frac{x}{2}\sqrt{\frac{a^4}{a^2 - b^2} - x^2} + \frac{a^4}{2(a^2 - b^2)}\sin^{-1}\frac{x\sqrt{a^2 - b^2}}{a^2}\right]_0^a$$

$$= 2\pi b^2 + \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\sin^{-1}\frac{\sqrt{a^2 - b^2}}{a}$$

E20) The loop is between $t = -\sqrt{3}$ and $t = \sqrt{3}$. Because of symmetry, it is enough to consider the curve between $t = 0$ and $t = \sqrt{3}$.

$$\therefore S = 2\pi \int_0^{\sqrt{3}} \left(t - \frac{t^3}{3}\right)\sqrt{4t^2 + (1 - t^2)^2}\, dt$$

$$= 2\pi \int_0^{\sqrt{3}} \left(t + \frac{2t^3}{3} - \frac{t^5}{3}\right) dt$$

$$= 2\pi \left[\frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{18}\right]_0^{\sqrt{3}}$$

$$= 3\pi.$$

E21) $S = 2\pi \int_0^{2\pi} a(1 - \cos\theta) \sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta}\, d\theta$

$= 4\pi a^2 \int_0^{2\pi} (1 - \cos\theta) \sin\frac{\theta}{2}\, d\theta$

$= 8\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2}\, d\theta$

$= 16\pi a^2 \int_0^{\pi} \sin^3\phi\, d\phi$

$= 32\pi a^2 \int_0^{\pi/2} \sin^3\phi\, d\phi$

$= 32\pi a^2 \cdot \frac{2}{3}$

$= \frac{64\pi a^2}{3}$